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and Stability Investigation " (DSMSI-2023)
Conference Proceedings

Kyiv, Ukraine, December 19-21, 2023

During the period December 19-21, 2023 the traditional International Conference “Dynamical Systems Modeling and Stability Investigation” was held. The conference is dedicated to the 77th anniversary of the outstanding Ukrainian scientist professor Denys Khusainov

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1. **Computer applied mathematics.**
   - Qualitative research methods of differential, functional-differential and difference systems
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   - Specialized software tools and systems Software modeling systems.

2. **Mathematical foundations of information technologies.**
   - Management and optimization methods
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   - Fuzzy modeling and systems with uncertainty
   - Modeling methods in economics and ecology
   - Methods and means of specification of subject areas
   - Methods and means of describing software systems
   - Model and temporal formalisms of modeling dynamic systems

There were 41 papers submitted for peer-review to this conference. Out of these, 27 papers were accepted for this volume, 22 as regular papers and 5 as short papers.

CONTENTS

1. Computer applied mathematics

MODELLING
&
STABILITY

Equivalent transformation and weighted H_∞-optimization of linear descriptor systems

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Abstract

In this paper, the problem of generalized H_∞-control is investigated for a class of admissible descriptor systems with the non-zero initial vector. A generalized performance measure is used, which characterize the weighted damping level of the bounded external disturbances, as well as the initial disturbances caused by an unknown initial vector. The main result of this work is the application of a non-degenerate transformation of the class of systems under consideration, which allows the use well-known evaluation methods and the achievement of desired performance measures for conventional systems. An example of robust stabilization of a hydraulic system with three tanks is given.

Keywords [1]

Descriptor system, H_∞-control, LMI

1. Introduction

In modern control theory, great attention is paid to descriptor (differential-algebraic) systems, which are used in modeling the movement of objects in mechanics, robotics, energy, electrical engineering, economics, etc. (see, e.g., [1–5]). Equations of motion, inputs and outputs of controlled objects may contain uncertain elements (parameters, external disturbances, measurement inaccuracies, etc.), that cause the need to solve the problems of robust stabilization and minimize the impact of bounded disturbances on the quality of transient processes (H_∞-*optimization*). A typical performance measure in the H_∞-optimization problem for systems with the zero initial state is a damping level of external (exogenous) disturbances, which corresponds to the maximum value of the ratio for L_2-norms of controlled output and disturbances of the object. For the class of linear descriptor systems

$$E\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad x(0_-) = x_0, \tag{1}$$

this characteristic coincides with the H_∞-norm of the matrix transfer function

$$\|G_{zw}\|_\infty = \sup_{\omega \in \mathbf{R}} \sqrt{\lambda_{\max}(G^\top(-i\omega)G(i\omega))}, \quad G(\lambda) = C(\lambda E - A)^{-1}B + D,$$

where $x \in \mathbf{R}^n$, $z \in \mathbf{R}^k$ and $w \in \mathbf{R}^s$ are the vectors of state, controlled output and input, respectively, and E, A, B, C, D are constant matrices, $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a matrix. In practice, it is reasonable to use weighted performance measure for control systems of the form [6]

$$J = \sup_{(w,x_0)\in\mathbf{\Omega}} \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}, \quad \|z\|_Q^2 = \sum_{t=0}^{\infty} y_t^\top Q y_t, \quad \|w\|_P^2 = \sum_{t=0}^{\infty} w_t^\top P w_t, \tag{2}$$

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where $\mathbf{\Omega}$ is the set of admissible pairs (w, x_0) of the system for which $0 < \|w\|_P^2 + x_0^{\mathsf{T}} X_0 x_0 < \infty$, and $P = P^{\mathsf{T}} > 0$, $Q = Q^{\mathsf{T}} > 0$, $X_0 = E^{\mathsf{T}} H E$ with $H = H^{\mathsf{T}} > 0$ are given weight matrices (see also [7, 8]). When $x_0 \in \operatorname{Ker} E$, we denote J as J_0. It is obvious that $J_0 \le J$. In the case $P = I_s$ and $Q = I_l$, the performance measure J_0 coincides with the standard performance index $\|G_{zw}\|_\infty$ used in the H_∞ control theory.

The value of J characterizes the weighted level of suppression of external disturbances, as well as initial disturbances caused by the nonzero initial vector. By using weight coefficients in these performance criteria, we can establish priorities between the components of controlled output and the unknown disturbances in the control system. Moreover, both the external disturbances acting upon the system and the errors of measured output can be components of the unknown disturbances.

The available methods of the H_∞-control are based on the criteria of validity of the upper bounds for the corresponding performance criteria established in terms of matrix equations and linear matrix inequalities (LMI) [9–11]. For the class of linear descriptor systems, similar statements were established in [12–15]. For the available methods of H_∞-optimization of these systems, see, e.g., [3,5,12,14,16].

This paper proposes new methods for solving the generalized H_∞-control problem for linear descriptor systems with performance measure of the form (2) based on a nonsingular transformation of such systems into ordinary ones and the application of well-known methods of synthesis of static and dynamic controllers. As a result, in a number of cases, the corresponding control synthesis algorithms are based on the solution of LMIs without additional rank restrictions. In particular, the order of the desired dynamic controller in such synthesis algorithms may not exceed the rank of the coefficient matrix at the derivative of a state of the original system. Sufficiently effective tools have been created for solving LMIs, for example, the LMI Toolbox of Matlab computer software [17]. To solve LMIs with rank constraints, you can use the LMIRank tools of the Matlab [18] or the Solve Block of the Mathcad Prime software [19]. A distinctive feature of the obtained results compared to known results is the application of weighted performance measures, which provide new opportunities for achieving the desired characteristics of descriptor control systems.

Notations: I_n is the identity $n \times n$ matrix; $0_{n \times m}$ is the zero $n \times m$ matrix; $X = X^{\mathsf{T}} > 0\ (\ge 0)$ is a positive (nonnegative) definite symmetric matrix; $\sigma(A)$ is the spectrum of A; A^{-1} (A^{+}) is the inverse (pseudo-inverse) of A; $\operatorname{Ker} A$ is the kernel of A; W_A is the right null matrix of $A \in \mathbf{R}^{m \times n}$, that is, $AW_A = 0$, $W_A \in \mathbf{R}^{n \times (n-r)}$, $\operatorname{rank} W_A = n - r$, where $r = \operatorname{rank} A < n$ ($W_A = 0$ if $r = n$); $\|x\|$ is the Euclidean norm of x; $\|x\|_P$ is the weighted L_2-norm of a vector function $w(t)$; $\mathbf{C}^-$ ($\mathbf{C}^+$) is the open half-plane $\operatorname{Re} \lambda < 0$ ($\operatorname{Re} \lambda > 0$).

2. Definitions and auxiliary statements

Consider the descriptor system (1) with $\operatorname{rank} E = \rho < n$ and the performance measure (2). The system is said to be admissible if the pair of matrices $\{E, A\}$ is *regular, stable* and *impulse-free* [1], i.e., respectively, $\det F(\lambda) \not\equiv 0$, $\sigma(F) \subset \mathbf{C}^-$ and $\deg \det F(\lambda) = \rho$, where $\sigma(F)$ is the finite spectrum of the matrix pencil $F(\lambda) = A - \lambda E$.

The pair of matrices $\{E, A\}$ is regular if and only if there exist nonsingular matrices L and R that transform it to the canonical Weierstrass form [20]. System (1) is impulse-free if and only if [2]

$$\operatorname{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \rho. \tag{3}$$

Let $E = E_l E_r^{\mathsf{T}}$ be the skeletal decomposition of E, where E_l and E_r are $n \times \rho$ matrices of full rank ρ. Denote the corresponding orthogonal complements $E_l^{\perp}$ and $E_r^{\perp}$ such that

$$E_l^{\mathsf T} E_l^{\perp} = 0, \quad E_r^{\mathsf T} E_r^{\perp} = 0, \ \det \begin{bmatrix} E_l & E_l^{\perp} \end{bmatrix} \neq 0, \ \det \begin{bmatrix} E_r & E_r^{\perp} \end{bmatrix} \neq 0.$$

Let's perform a nonsingular transformation of system (1) based on the relations

$$LER = \begin{bmatrix} I_\rho & 0 \\ 0 & 0 \end{bmatrix}, \ LAR = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \ x = R \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ x_1 \in \mathbf{R}^{\rho}, \ x_2 \in \mathbf{R}^{n-\rho}, \tag{4}$$

$$L = \begin{bmatrix} E_l^{+} \\ E_l^{\perp +} \end{bmatrix}, \ E_l^{+} = (E_l^{\mathsf T} E_l)^{-1} E_l^{\mathsf T}, \quad E_l^{\perp +} = (E_l^{\perp \mathsf T} E_l^{\perp})^{-1} E_l^{\perp \mathsf T},$$

$$R = \begin{bmatrix} E_r^{+\mathsf T} & E_r^{\perp + \mathsf T} \end{bmatrix}, \ E_r^{+} = (E_r^{\mathsf T} E_r)^{-1} E_r^{\mathsf T}, \quad E_r^{\perp +} = (E_r^{\perp \mathsf T} E_r^{\perp})^{-1} E_r^{\perp \mathsf T}.$$

It is easy to establish that the equality (3) holds if and only if

$$\det A_4 \neq 0, \quad A_4 = E_l^{\perp +} A E_r^{\perp + \mathsf T}. \tag{5}$$

Using (4) and (5), we have $x_2 = -A_4^{-1}(A_3 x_1 + B_2 w)$, and the dynamics of impulse-free system (1) is described as

$$\dot{x}_1 = \bar{A} x_1 + \bar{B} w, \quad z = \bar{C} x_1 + \bar{D} w, \ x_1(0) = x_{10}, \tag{6}$$

$$\bar{A} = A_1 - A_2 A_4^{-1} A_3, \quad \bar{B} = B_1 - A_2 A_4^{-1} B_2, \quad \bar{C} = C_1 - C_2 A_4^{-1} A_3, \quad \bar{D} = D - C_2 A_4^{-1} B_2,$$

$$LB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ CR = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

In this case, the spectrum of $\bar{A}$ coincides with $\sigma(F)$, since

$$\begin{bmatrix} I_\rho & -A_2 A_4^{-1} \\ 0 & I_{n-\rho} \end{bmatrix} LF(\lambda) R \begin{bmatrix} I_\rho & 0 \\ -A_4^{-1} A_3 & I_{n-\rho} \end{bmatrix} = \begin{bmatrix} \bar{A} - \lambda I_\rho & 0 \\ 0 & A_4 \end{bmatrix}$$

and performance measure J of the system does not depend on x_2. Indeed, since $L^{-1} = \begin{bmatrix} E_l & E_l^{\perp} \end{bmatrix}$, we have

$$x_0^{\mathsf T} X_0 x_0 = \begin{bmatrix} x_{10}^{\mathsf T} & x_{20}^{\mathsf T} \end{bmatrix} R^{\mathsf T} E^{\mathsf T} L^{\mathsf T} L^{-1\mathsf T} H L^{-1} LER \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = x_{10}^{\mathsf T} \bar{H} x_{10},$$

where $\bar{H} = E_l^{\mathsf T} H E_l$. Therefore, applying [21, lemma 2.3] to system (6), we have the following statement.

Lemma 1. *System* (1) *is admissible and* $J_0 < \gamma$ *if and only if there exists a matrix* $X = X^{\mathsf T} > 0$ *such that*

$$\bar{\Phi}(X) = \begin{bmatrix} \bar{A}^{\mathsf T} X + X\bar{A} + \bar{C}^{\mathsf T} Q \bar{C} & X\bar{B} + \bar{C}^{\mathsf T} Q \bar{D} \\ \bar{B}^{\mathsf T} X + \bar{D}^{\mathsf T} Q \bar{C} & \bar{D}^{\mathsf T} Q \bar{D} - \gamma^2 P \end{bmatrix} < 0. \tag{7}$$

The system is admissible and $J < \gamma$ *if and only if the LMIs* (7) *and*

$$0 < X < \gamma^2 \bar{H} \tag{8}$$

are feasible.

Lemma 1 can be used to calculate the characteristics J_0 and J of system (1) based on solving the corresponding optimization problems. At the same time, the restrictions in these problems are used exclusively in terms of LMIs. For example,

$$J = \inf \left\{ \gamma : \bar{\Phi}(X) < 0, 0 < X < \gamma^2 \bar{H} \right\}.$$

The perturbation vector $w(t)$ and the initial vector x_0 in system (1) are *worst* with respect to J, if in (2) the supremum is reached, i.e. $\|z\|_Q^2 = J^2(\|w\|_P^2 + x_0^{\mathsf T} X_0 x_0)$. Methods of finding such vectors in individual cases are proposed in [7, 22, 23]. For example, if system (1) is admissible and there exists a matrix X such that

$$A_0^{\mathsf T} X + X^{\mathsf T} A_0 + X^{\mathsf T} R_0 X + Q_0 = 0, \quad 0 < E^{\mathsf T} X = X^{\mathsf T} E \le \gamma^2 X_0,$$

where

$$A_0 = A + BR_1^{-1} D^{\mathsf T} QC,\ R_0 = BR_1^{-1} B^{\mathsf T},\ Q_0 = C^{\mathsf T}(Q + QDR_1^{-1} D^{\mathsf T} Q)C,\ R_1 = \gamma^2 P - D^{\mathsf T} QD > 0$$

and $\gamma = J$, then the structured vector of external disturbances in the form of linear state feedback

$$w = K_0 x, \quad K_0 = R_1^{-1}(B^{\mathsf T} X + D^{\mathsf T} QC), \tag{9}$$

and the arbitrary initial vector $x_0 \in \mathrm{Ker}(E^{\mathsf T} X - \gamma^2 X_0)$ are the worst with respect to J for system (1) [23].

We present another method of finding the worst pair $\{w, x_0\}$ with respect to J for impulse-free system (1) based on the transformation (4). Under condition (5), we construct the worst initial vector in the form

$$x_0 = \mathrm{R}\begin{bmatrix} x_{10} \\ -A_4^{-1}(A_3 x_{10} + B_2 w(0+)) \end{bmatrix}, \tag{10}$$

where $\{w, x_{10}\}$ is the worst pair of system (6) with respect to J. According to the Schur complement lemma, the condition (7) is equivalent to the Riccati matrix inequality

$$\bar A_0^{\mathsf T} X + X^{\mathsf T} \bar A_0 + X^{\mathsf T} \bar R_0 X + \bar Q_0 < 0, \tag{11}$$

where

$$\bar A_0 = \bar A + \bar B \bar R_1^{-1} \bar D^{\mathsf T} Q \bar C,\ R_0 = BR_1^{-1} B^{\mathsf T},\ \bar Q_0 = \bar C^{\mathsf T}(Q + Q\bar D \bar R_1^{-1} \bar D^{\mathsf T} Q)\bar C,\ \bar R_1 = \gamma^2 P - \bar D^{\mathsf T} Q \bar D > 0 .$$

If the pair $\{\bar A, \bar B\}$ is controllable, the pair $\{\bar A, \bar C\}$ is observable and $J_0 < \gamma$, then the corresponding Riccati matrix equation

$$\bar A_0^{\mathsf T} X + X^{\mathsf T} \bar A_0 + X^{\mathsf T} \bar R_0 X + \bar Q_0 = 0 \tag{12}$$

has the solutions X_- and X_+ such that $\sigma(\bar A_0 + \bar R_0 X_\pm) \subset \mathbf{C}^\pm$, $0 < X_- < X_+$, and every solution of inequality (11) belongs to the interval $X_- < X < X_+$ (see [24, 25]). Moreover, if $J < \gamma$ $(J \le \gamma)$ and X satisfies (12), then $X < \gamma^2 \bar H$ $(X \le \gamma^2 \bar H)$. Indeed, setting $v(x_1) = x_1^{\mathsf T} X x_1$ and

$$w = \bar K_0 x_1,\ \bar K_0 = \bar R_1^{-1}(\bar B^{\mathsf T} X + \bar D^{\mathsf T} Q \bar C), \tag{13}$$

we get the equality $\dot v(x_1) + z^{\mathsf T} Q z - \gamma^2 w^{\mathsf T} P w = 0$, where $\dot v(x_1)$ is the derivative of the function $v(x_1)$ due to system (6). After integrating this equality over an infinite interval under the condition $J < \gamma$ we get $\|z\|_Q^2 - \gamma^2 \|w\|_P^2 = x_{10}^{\mathsf T} X x_{10} < \gamma^2 x_{10}^{\mathsf T} \bar H x_{10}$ for any $x_{10} \ne 0$, otherwise $J \ge \gamma$. If $J = \gamma$ then under conditions (12) and (13) the equality $x_{10}^{\mathsf T} X x_{10} = \gamma^2 x_{10}^{\mathsf T} \bar H x_{10}$ or its equivalent $(X - \gamma^2 \bar H) x_{10} = 0$ is possible for some $x_{10} \ne 0$. At the same time, $\|z\|_Q^2 = J^2(\|w\|_P^2 + x_{10}^{\mathsf T} \bar H x_{10})$, i.e. in (2) the supremum is reached. Hence, the following statement holds.

Lemma 2. Let $X = X_- > 0$ *be the stabilizing solution of the Riccati equation* (12) *under the conditions* (5) *and* $\gamma = J$. *Then the structured vector of external disturbances* (13), *where* $x_1 = x_1(t, x_{10})$ *is a solution of the system* $\dot x_1 = (\bar A + \bar B \bar K_0) x_1$, *and the arbitrary vector* (10) *at* $x_{10} \in \mathrm{Ker}(X - J^2 \bar H)$ *are worst with respect to* J *in system* (1).

3. Main results

Consider a descriptor control system

$$E\dot x = Ax + B_1 w + B_2 u, \quad x(0_-) = x_0,$$

$$z = C_1 x + D_{11} w + D_{12} u, \tag{14}$$
$$y = C_2 x + D_{21} w + D_{22} u,$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $w \in \mathbf{R}^s$, $z \in \mathbf{R}^k$ and $y \in \mathbf{R}^l$ are the vectors of state, control, external disturbances, controlled and observed outputs, respectively. In (14), all matrix coefficients are constant, $\operatorname{rank} E = \rho \le n$ and the pair $\{E, A\}$ is regular and impulse-free. We are interested in the stabilizing control laws that guarantee the asymptotic stability of the closed loop system at $w \equiv 0$ and the desired estimate $J < \gamma$ of performance measure (2) of the system with respect to the controlled output z. Static and dynamic controllers that minimize the performance measure J are called J*-optimal*. For the identity weight matrices P and Q, the J_0-optimal control is called *H∞-optimal*. The search for J_0- and J-optimal controllers can be performed on the basis of achieving the corresponding estimates $J_0 < \gamma$ and $J < \gamma$ at the minimum possible value of γ.

We apply the equivalent transformation (4) to system (14). Excluding the variable

$$x_2 = -A_4^{-1}(A_3 x_1 + B_{12} w + B_{22} u)$$

under condition (5), we get the usual system

$$\dot{x}_1 = \bar{A} x_1 + \bar{B}_1 w + \bar{B}_2 u, \quad x_1(0) = x_{10},$$
$$z = \bar{C}_1 x_1 + \bar{D}_{11} w + \bar{D}_{12} u, \tag{15}$$
$$y = \bar{C}_2 x + \bar{D}_{21} w + \bar{D}_{22} u,$$

where

$$\bar{A} = A_1 - A_2 A_4^{-1} A_3, \ \bar{B}_1 = B_{11} - A_2 A_4^{-1} B_{12}, \ \bar{B}_2 = B_{21} - A_2 A_4^{-1} B_{22},$$
$$\bar{C}_1 = C_{11} - C_{12} A_4^{-1} A_3, \ \bar{D}_{11} = D_{11} - C_{12} A_4^{-1} B_{12}, \ \bar{D}_{12} = D_{12} - C_{12} A_4^{-1} B_{22},$$
$$\bar{C}_2 = C_{21} - C_{22} A_4^{-1} A_3, \ \bar{D}_{21} = D_{21} - C_{22} A_4^{-1} B_{12}, \ \bar{D}_{22} = D_{22} - C_{22} A_4^{-1} B_{22},$$
$$LB_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \ LB_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \ C_1 R = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \ C_2 R = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}.$$

Defining the performance measure (2) for this system, we use the expression $x_0^{\mathsf{T}} X_0 x_0 = x_{10}^{\mathsf{T}} \bar{H} x_{10}$, where $\bar{H} = E_l^{\mathsf{T}} H E_l$ (see previous section). Thus, the J_0- and J- optimization problems under condition (5) are reduced to application of well-known methods for solving similar problems to system (15) [21,26].

3.1. Static controller

When using the static output-feedback controller

$$u = Ky, \quad \det(I_m - K\bar{D}_{22}) \ne 0, \tag{16}$$

for system (15), the closed loop system has the form

$$\dot{x}_1 = A_* x_1 + B_* w, \quad z = C_* x_1 + D_* w, \tag{17}$$

where $A_* = \bar{A} + \bar{B}_2 K_0 \bar{C}_2$, $B_* = \bar{B}_1 + \bar{B}_2 K_0 \bar{D}_{21}$, $C_* = \bar{C}_1 + \bar{D}_{12} K_0 \bar{C}_2$, $D_* = \bar{D}_{11} + \bar{D}_{12} K_0 \bar{D}_{21}$ and $K_0 = (I_m - K\bar{D}_{22})^{-1} K$. Applying the Schur complement lemma, we rewrite the matrix inequality (7) in Lemma 1 for system (17) as LMI with respect to K_0:

$$\begin{bmatrix} A_*^{\mathsf{T}} X + X A_* & X B_* & C_*^{\mathsf{T}} \\ B_*^{\mathsf{T}} X & -\gamma^2 P & D_*^{\mathsf{T}} \\ C_* & D_* & -Q^{-1} \end{bmatrix} = L_0^{\mathsf{T}} K_0 R_0 + R_0^{\mathsf{T}} K_0^{\mathsf{T}} L_0 + \Omega < 0, \tag{18}$$

where

$$R_0=\begin{bmatrix}\bar C_2 & \bar D_{21} & 0_{l\times k}\end{bmatrix},\ L_0=\begin{bmatrix}\bar B_2^{\mathsf T}X & 0_{m\times s} & \bar D_{12}^{\mathsf T}\end{bmatrix},\ \Omega=\begin{bmatrix}\bar A^{\mathsf T}X+X\bar A & X\bar B_1 & \bar C_1^{\mathsf T}\\ B_1^{\mathsf T}X & -\gamma^2P & \bar D_{11}^{\mathsf T}\\ \bar C_1 & \bar D_{11} & -Q^{-1}\end{bmatrix}.$$

Based on Theorems 4.3 and 4.4 from [26], we have the following result.

Theorem 1. *For system* (14) *there exists a static output-feedback controller* (16), *for which the closed loop system is admissible and* $J<\gamma$, *if and only if the relations* (8) *and*

$$W_{\bar R}^{\mathsf T}\begin{bmatrix}\bar A^{\mathsf T}X+X\bar A+\bar C_1^{\mathsf T}Q\bar C_1 & X\bar B_1+\bar C_1^{\mathsf T}Q\bar D_{11}\\ \bar B_1^{\mathsf T}X+\bar D_{11}^{\mathsf T}Q\bar C_1 & \bar D_{11}^{\mathsf T}Q\bar D_{11}-\gamma^2P\end{bmatrix}W_{\bar R}<0, \tag{19}$$

$$W_{\bar L}^{\mathsf T}\begin{bmatrix}\bar AY+Y\bar A^{\mathsf T}+\bar B_1P^{-1}\bar B_1^{\mathsf T} & Y\bar C_1^{\mathsf T}+\bar B_1P^{-1}\bar D_{11}^{\mathsf T}\\ \bar C_1Y+\bar D_{11}P^{-1}\bar B_1^{\mathsf T} & \bar D_{11}P^{-1}D_{11}^{\mathsf T}-\gamma^2Q^{-1}\end{bmatrix}W_{\bar L}<0, \tag{20}$$

$$W=\begin{bmatrix}X & \gamma I_\rho\\ \gamma I_\rho & Y\end{bmatrix}\ge 0,\quad \operatorname{rank}W=\rho, \tag{21}$$

where $\bar R=\begin{bmatrix}\bar C_2 & \bar D_{21}\end{bmatrix}$ *and* $\bar L=\begin{bmatrix}\bar B_2^{\mathsf T} & \bar D_{12}^{\mathsf T}\end{bmatrix}$, *are feasible. Then gain matrix of this controller can be constructed in the form* $K=K_0(I_l+\bar D_{22}K_0)^{-1}$, *where* K_0 *is a solution of the LMI* (18).

We present the following corollary of Lemma 1 and Theorem 1 under additional conditions

$$\bar C_2=I_l,\ \bar D_{21}=0,\ \bar D_{22}=0,\ \bar D_{11}^{\mathsf T}Q\bar D_{11}<\gamma^2P. \tag{22}$$

Corollary 1. *The following statements are equivalent:*

1) *for system* (14), *there is a static state-feedback controller* $u=Kx$ *for which the closed loop system is admissible and* $J<\gamma$;
2) *there is a matrix* $Y>\bar H^{-1}$ *that satisfies the LMI (20);*
3) *there exist matrices* $Y>\bar H^{-1}$ *and* Z *satisfying the LMI*

$$\begin{bmatrix}\gamma^2(\bar AY+Y\bar A^{\mathsf T}+\bar B_2Z+Z^{\mathsf T}\bar B_2^{\mathsf T} & \gamma^2\bar B_1 & Y\bar C_1^{\mathsf T}+Z^{\mathsf T}\bar D_{12}^{\mathsf T}\\ \gamma^2\bar B_1^{\mathsf T} & -\gamma^2P & \bar D_{11}^{\mathsf T}\\ C_* & \bar D_{11} & -Q^{-1}\end{bmatrix}<0. \tag{23}$$

When statement 3 *holds, the gain matrix can be found in the form* $K=ZY^{-1}$.

The equivalence of statements 1 and 2 in Corollary 1 follows from Theorem 1, since $W_{\bar R}=\begin{bmatrix}0 & I_s\end{bmatrix}^{\mathsf T}$ under the conditions (22) and the inequality (19) does not depend on X. Matrix inequality (23) in statement 3 is a corollary of multiplying the first block row on the left and the first block column on the right in (18) by $Y=\gamma^2X^{-1}$ under the conditions (22) and $K=ZY^{-1}$.

Remark 1. If the condition (5) is not holds and there is a matrix K_* such that $\det\left[E_l^{\perp+}(A+B_2K_*C_2)E_r^{\perp+\mathsf T}\right]\ne 0$, then in the case $D_{22}=0$, we can apply instead of (16) the controller $u=K_*y+\tilde u$, where $\tilde u=Ky$ is a new control that solves the considered problem for the closed loop system.

3.2. Dynamic controller

When using the dynamic controller of the order p

$$\dot\xi=Z\xi+Vy,\quad u=U\xi+Ky,\ \xi(0)=0, \tag{24}$$

for system (15), the closed loop system in an extended phase space $\mathbf R^{\rho+p}$ has the form

$$\dot{\hat x}=\hat A_*\hat x+\hat B_*w,\quad z=\hat C_*\hat x+\hat D_*w,\ \hat x(0)=\hat x_0, \tag{25}$$

where

$$\widehat{A}_* = \widehat{A} + \widehat{B}_2\widehat{K}_0\widehat{C}_2,\ \ \widehat{B}_* = \widehat{B}_1 + \widehat{B}_2\widehat{K}_0\widehat{D}_{21},\ \ \widehat{C}_* = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_0\widehat{C}_2,\ \ \widehat{D}_* = \widehat{D}_{11} + \widehat{D}_{12}\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{x} = \begin{bmatrix} x_1 \\ \xi \end{bmatrix},\ \widehat{A} = \begin{bmatrix} \bar{A} & 0_{\rho\times p} \\ 0_{p\times\rho} & 0_{p\times p} \end{bmatrix},\ \widehat{B}_1 = \begin{bmatrix} \bar{B}_1 \\ 0_{p\times s} \end{bmatrix},\ \widehat{B}_2 = \begin{bmatrix} \bar{B}_2 & 0_{\rho\times p} \\ 0_{p\times m} & I_p \end{bmatrix},$$

$$\widehat{C}_1 = \begin{bmatrix} \bar{C}_1 & 0_{k\times p} \end{bmatrix},\ \widehat{D}_{11} = \bar{D}_{11},\ \widehat{D}_{12} = \begin{bmatrix} \bar{D}_{12} & 0_{k\times p} \end{bmatrix},\ \widehat{C}_2 = \begin{bmatrix} \widehat{C}_2 & 0_{l\times p} \\ 0_{p\times\rho} & I_p \end{bmatrix},\ \widehat{D}_{21} = \begin{bmatrix} \bar{D}_{21} \\ 0_{p\times s} \end{bmatrix},$$

We define a performance measure $\widehat{J}$ for system (25) of the form (2) with weight matrices P, Q and $\widehat{X}_0$, , where $\widehat{X}_0$ is some block $(\rho+p)\times(\rho+p)$ matrix, whose first diagonal block is $\bar{H}$. Since the initial vector of the controller (24) is zero, the value of $\widehat{J}$ coincides with J.

Lemma 3 ([26]). *Given positive definite matrices* $X, Y \in \mathbf{R}^{\rho\times\rho}$ *and the number* $\gamma > 0$, *there exist matrices* $X_1, Y_1 \in \mathbf{R}^{p\times\rho}$ *and* $X_2, Y_2 \in \mathbf{R}^{p\times p}$ *such that*

$$\widehat{X} = \begin{bmatrix} X & X_1^{\mathsf{T}} \\ X_1 & X_2 \end{bmatrix} > 0,\ \ \widehat{Y} = \begin{bmatrix} Y & Y_1^{\mathsf{T}} \\ Y_1 & Y_2 \end{bmatrix} > 0,\ \ \widehat{X}\widehat{Y} = I_{\rho+p}, \tag{26}$$

if and only if

$$W = \begin{bmatrix} X & \gamma I_\rho \\ \gamma I_\rho & Y \end{bmatrix} \geq 0, \quad \text{rank}\, W = \rho + p. \tag{27}$$

Theorem 2. *For system* (14), *there is a dynamic controller* (24) *of order* $p \leq \rho$, *for which the closed loop system is admissible and J < γ, if and only if the system of relations* (8), (19), (20) *and* (27) *is feasible with respect to* X *and* Y. *Matrices of such controller can be defined in the form*

$$\begin{bmatrix} K & U \\ V & Z \end{bmatrix} = (I_{m+p} + \widehat{K}_0\widehat{D}_{22})^{-1}\widehat{K}_0,\ \ \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix},\ \ \widehat{D}_{22} = \begin{bmatrix} \bar{D}_{22} & 0_{l\times p} \\ 0_{p\times m} & 0_{p\times p} \end{bmatrix}, \tag{28}$$

where $\widehat{K}_0$ *is a solution of the LMI*

$$\widehat{L}^{\mathsf{T}}\widehat{K}_0\widehat{R} + \widehat{R}^{\mathsf{T}}\widehat{K}_0^{\mathsf{T}}\widehat{L} + \widehat{\Omega} < 0, \tag{29}$$

where

$$\widehat{R} = \begin{bmatrix} \widehat{C}_2 & \widehat{D}_{21} & 0_{(l+p)\times k} \end{bmatrix},\ \widehat{L} = \begin{bmatrix} \widehat{B}_2^{\mathsf{T}}\widehat{X} & 0_{(m+p)\times s} & \widehat{D}_{12}^{\mathsf{T}} \end{bmatrix},\ \widehat{\Omega} = \begin{bmatrix} \widehat{A}^{\mathsf{T}}\widehat{X} + \widehat{X}\widehat{A} & \widehat{X}\widehat{B}_1 & \widehat{C}_1^{\mathsf{T}} \\ \widehat{B}_1^{\mathsf{T}}X & -\gamma^2 P & \widehat{D}_{11}^{\mathsf{T}} \\ \widehat{C}_1 & \widehat{D}_{11} & -Q^{-1} \end{bmatrix}.$$

The block matrix $\widehat{X}$ *in* (29) *is formed on the basis of Lemma 3 according to* (26), *where* X *and* Y *satisfy* (8), (19), (20) *and* (27).

Note that Theorems 1 and 2 without using the constraint $X < \gamma^2\bar{H}$ give the existence criteria and methods for constructing stabilizing controllers that provide the estimate $J_0 < \gamma$ for the corresponding closed loop systems. The conditions of Theorem 2 in the case $p = 0$ are a criterion for the existence of a static controller (16) with the specified properties in Theorem 1. Construction of dynamic controllers of the order $p = \rho$ satisfying Theorem 2 reduces to the solution of the LMI system without additional restrictions. In this case, the rank restriction in (27) holds automatically.

We present the following algorithm for constructing a dynamic controller (24), which satisfies Theorem 2.

Algorithm 1.

1) Calculation of transformation matrices (4) and coefficient matrices of system (15);
2) calculation of $W_{\bar{R}}$ and $W_{\bar{L}}$ with $\bar{R} = \begin{bmatrix} \bar{C}_2 & \bar{D}_{21} \end{bmatrix}$, $\bar{L} = \begin{bmatrix} \bar{B}_2^{\mathsf{T}} & \bar{D}_{12}^{\mathsf{T}} \end{bmatrix}$;
3) finding matrices X and Y that satisfy (8), (19), (20) and (27);

4) construction of the decomposition $\Delta = Y - \gamma^2 X^{-1} = S^{\mathsf{T}} S \geq 0,$ where $S \in \mathbf{R}^{p \times \rho}$, $\operatorname{Ker} S = \operatorname{Ker} \Delta,$ and formation of the block matrix

$$\widehat{X} = \begin{bmatrix} X & X_1^{\mathsf{T}} \\ X_1 & X_2 \end{bmatrix} > 0,\ X_1 = \frac{1}{\gamma} SX,\ X_2 = \frac{1}{\gamma^2} SXS^{\mathsf{T}} + I_p;$$

5) solution of the LMI (29) with respect to $\widehat{K}_0$ under the condition $\det(I_m + K_0 \bar{D}_{22}) \neq 0;$

6) calculation of the controller matrices according to (28).

This algorithm can be implemented, for example, by means of the Matlab software. If in the point 4 of the algorithm $\Delta = 0,$ i.e. $\operatorname{rank} W = \rho,$ then solving the LMI (18), we obtain the static controller (16), which satisfies Theorem 1.

4. Example

Consider a linearized model of hydraulic system with three tanks connected in series. This model is described in the form of a descriptor control system (14) with the following matrices [27]:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ D_{11} = 0_{1\times 2},\ D_{12} = 1,\ D_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ D_{22} = 0_{2\times 1},$$

Components of the state vector $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathsf{T}}$ determine the liquid levels in the corresponding tanks, the vector $w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^{\mathsf{T}}$ is formed by the uncontrolled disturbances w_1 and the error w_2 of measurements $y = \begin{bmatrix} x_1 & x_2 + w_2 \end{bmatrix}^{\mathsf{T}}$ and the controlled output is $z = x_3 + u.$ The role of control u regulating the level of liquid in the first two tanks is played by the debit (flow) of liquid through the pump from the third tank into the first tank (see Fig. 1).

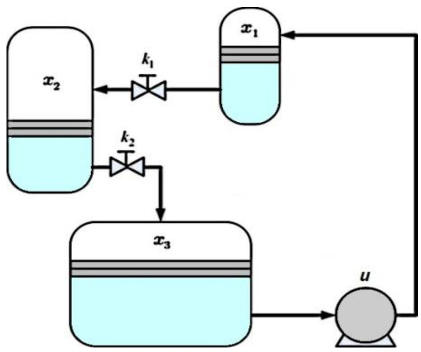

Figure 1: Hydraulic system with three tanks

Let the weight matrices of performance measure (2) and admissible parameter values be

$$P = \operatorname{diag}\{2,1\},\ Q = 1,\ X_0 = \operatorname{diag}\{3,2,0\},\ H = \operatorname{diag}\{3,2,1\},\ k_1 = 1,\ k_2 = 1.5.$$

In this example $n = 3,\ m = k = 1,\ l = s = 2,\ \rho = 2,$ a pair of matrices $\{E, A\}$ is admissible, and the system (14) is impulse-controllable and impulse-observable [28]. The system without control has the performance measure $J = 1.17851.$ Using the LMIRank tools of the Matlab software for solving LMIs with rank constraints [18] based on Theorem 1 at $\gamma = 1,$ we found the static controller matrix

$$K = -[0.68632 \quad 0.39429]$$

such that the closed loop system (17) is admissible and $J = 0.97981 < \gamma$. At the same time, its finite spectrum coincides with $\sigma(A_*) = \{-1.59316 \pm 0.62098\,\mathrm{i}\}$, where A_* is a matrix of system (17).

Applying Lemma 2, the worst perturbation and the worst initial vector with respect to J for the closed loop system was found in the form

$$w = \bar{K}_0 x_1, \quad \bar{K}_0 = \begin{bmatrix} 0.51303 & 1.12615 \\ 0.20056 & -0.23325 \end{bmatrix}, \tag{30}$$

$$x_0 = [-0.80818 \quad -0.58894 \quad 1.39712]^{\mathrm{T}}. \tag{31}$$

In Fig. 2 shows the behavior of the solution of the closed loop system under the worst conditions (30) and (31), and in Fig. 3 shows the vector-function (30) of the worst disturbance.

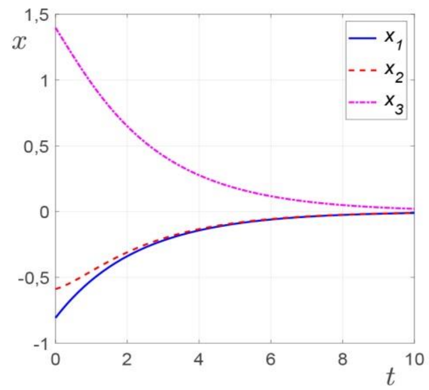

Figure 2: Behavior of a closed loop system

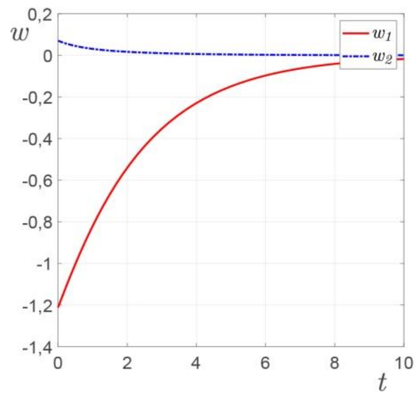

Figure 3: The worst perturbation with respect to J

Next, applying Algorithm 1 the matrices of the approximate J-optimal dynamic controller (24) of order $p = \rho$ are found for system (14):

$$Z = \begin{bmatrix} -0.64929 & -0.00005 \\ 0.00001 & -0.93121 \end{bmatrix}, \quad V = \begin{bmatrix} 0.00061 & 0.00030 \\ 0.00273 & -0.00245 \end{bmatrix},$$

$$U = -[0.00300 \quad 0.00154], \quad K = -[1.09073 \quad 0.04547],$$

for which the closed loop system is admissible with the finite spectrum

$$\{-1.99973, \ -1.59100, \ -0.93121, \ -0.64929\}$$

and the minimum value of performance measure $J = 0.97895$.

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Well-posedness of a Dirichlet problem for an integro-differential equation with a small time derivative coefficient

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Abstract

We use a priory inequalities method and provide results on well-posendess of Dirichlet initial-boundary value problem for the integro-differential equation with Volterra-type integral terms and small coefficient near time derivative. The existence and uniqueness of generalized solution is proved.

Keywords [2]

A priory estimations, well-posedness, Dirichlet problem, integro-differential equation, Volterra operator, generalized solutions

1. Introduction

The field of partial differential equations (PDEs) is extensive and widely recognized in mathematics. Its applications are diverse, playing a crucial role in addressing real-life challenges in fields such as economics, biology, chemistry, physics, and more. It should be noted that it serves as a powerful tool for researchers engaged in mathematical modeling.

In systems described by partial differential equations, the behavior at any given moment depends solely on the current state of the object. However, in certain scenarios, the future development of a process may also rely on preceding moments (history of the process), leading to the inclusion of integral terms in the corresponding models. A significant subset of equations considering the history of a process is the class of integro-differential equations with partial derivatives of the Volterra type [1]. There are exist numerous valuable contributions to this area, e.g. [2].

S.I. Lyashko has made substantial contributions to the understanding of well-posed problems, optimal control, and controllability in processes governed by various types of PDEs. His development and use of a priori inequalities in negative norms method has yielded notable results, documented in works like [3], [4], [5], [6], and others cited in the bibliography therein.

The fundamental principles of the theory of a priori inequalities in negative norms, along with selected applications, are expounded in the classic monograph [7] and, to some extent, in works [8], [9]. The methodology developed by S.I. Lyashko and collaborators has proven effective in investigating well-posedness in problem formulations, optimal control, and controllability in systems with distributed parameters. While a comprehensive bibliography on these topics is beyond the scope here, we reference a few key works: [5], [10], [11], [12], [13], [14], [15], [16], [17], [18].

Interestingly, the aforementioned approach extends to Dirichlet problems for integro-differential equations featuring Volterra-type integral components. Investigations into hyperbolic-type integro-differential equations are detailed in works [19], [20], elliptic equations in [21], and parabolic types in [22], [23]. Special equations with Volterra-type integral components find consideration in [24], [25]. The monograph [26] serves as a compilation of results in this area.

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In our work we investigate a Dirichlet problem with homogeneous initial-boundary conditions for the integro-differential equation with the following operator with a small time derivative coefficient:

$$\mathcal{L}_\varepsilon u = -\varepsilon \frac{\partial u}{\partial t} + u - \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(K_i(t,\tau,x)\frac{\partial u(x,\tau)}{\partial x_i}\right) d\tau.$$

In the work [25] the following equation was considered

$$u(x,t) - \int_0^t \sum_{i=1}^n K_i\,(t,\tau)\frac{\partial^2 u(x,\tau)}{\partial x_i^2} d\tau = f(x,t).$$

In the cited paper work the kernel $K(t,\tau)$ must be the same for all second derivatives and it is considered to be non-negative

$$\int_0^T \int_0^T K\,(t,\tau) u(t) u(\tau) d\tau dt \geq 0, \forall u \in L_2([0,T]),$$

which is quite a limitation. In works [26],[27] the well-posedness of a Dirichlet problem was proved for the equation

$$u(x,t) - \sum_{i=1}^n \int_0^t K\,(t,\tau) u_{x_i x_i}(x,\tau) + L_i(x,t,\tau) u_{x_i}(x,\tau) d\tau - \int_0^t R\,(x,t,\tau) u(x,\tau) d\tau = f(x,t).$$

The result was obtained for a bigger class of kernels, which should be non-negative only at the diagonal

$$K(t,t) \geq 0, \forall t \in [0,T].$$

However, in these works there was a strong demanding condition of using only one kernel, the same for all the second derivatives.

In our work we suggest an approach, which allows us to prove a priori estimations in the case when different kernels correspond to different second derivatives. Also, we need kernels to be non-negative only on the diagonal. Moreover, in our work kernels can depend on spatial variable x.

In doing so, we expand the range of equations to which the method of a priori inequalities can be effectively applied. These inequalities, as established, serve as a basis for justifying the well-posedness of the initial-boundary value problem. An inherent advantage of the proposed approach lies in the potential applicability of these inequalities to various real-life scenarios. For instance, they can be utilized in future applications such as the formulation of theorems concerning the existence of optimal control. In this context, control is executed through diverse types of operators, utilizing the right-hand side of equation investigated equation. These control operators operate within the spaces of generalized functions (distributions), enabling the modeling of impulse-point actions on the system. Additionally, the established inequalities can contribute to the development of numerical methods for assessing generalized solutions and optimal control, as well as theorems pertaining to their convergence. These properties emerge as straightforward consequences of the overarching general theory [7] and the specific inequalities obtained in our paper.

2. Spaces and general notations

Let us consider the following operator

$$\mathcal{L}_\varepsilon u = -\varepsilon \frac{\partial u}{\partial t} + u - \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(K_i(t,\tau,x)\frac{\partial u(x,\tau)}{\partial x_i}\right) d\tau, \tag{1}$$

and the corresponding adjoin operator

$$\mathcal{L}_\varepsilon^* u = \varepsilon \frac{\partial u}{\partial t} + u - \int_t^T \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(K_i(t,\tau,x)\frac{\partial u(x,\tau)}{\partial x_i}\right) d\tau, \tag{2}$$

where ε is some positive number. The main result of the work is to show that for some small enough values of parameter ε, corresponding initial and boundary problems would be well-posed. We consider cylindrical domain $Q = \Omega \times (0,T)$, where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. By C_{BR}^∞ we denote the set of all smooth (infinitely differentiable) in Q functions that satisfy conditions

$$u|_{\partial\Omega} = 0 \text{ and } u|_{t=T} = 0.$$

For now let us define the domain of the operator $\mathcal{L}_\varepsilon$ to be C^∞_{BR}. In the same way we define C^∞_{BR*} to be the set of all smooth (infinitely differentiable) in Q functions that satisfy conditions

$$v|_{\partial\Omega} = 0 \text{ and } v|_{t=0} = 0$$

and $\mathcal{L}^*_\varepsilon$ is defined on C^∞_{BR*}.We assume as well that the kernels $K_i(t,\tau,x)$ meet the following conditions:

- $K_i(t,\tau,x)$ are continuously-differentiable with respect to the variables t, τ, x (by K_i' we denote the derivative of the function K_i with respect to the variable t);
- there exists $\alpha > 0$, such that for all $i = 1,2,\dots,n$, $t \in [0,T]$, $x \in \Omega$ the following inequality holds

$$K_i(t,t,x) \geq \alpha. \tag{3}$$

We assume that the parameter ε is small enough to satisfy the condition

$$\varepsilon \in \left(0; \frac{\alpha^2}{2TM^2}\right). \tag{4}$$

Here and further M is a positive constant greater then all K_i and K'_i for all admissible t,τ,x.

By E, E_+, W, and W_+ we will denote completion of spaces C^∞_{BR}, C^∞_{BR*}, C^∞_{BR}, and C^∞_{BR*} respectively, by the following norms

$$\begin{aligned} ||u||^2_E &= ||u||^2_{E_+} = ||u||^2_{L_2(Q)} + ||u_t||^2_{L_2(Q)} + \sum_{i=1}^n \left\|u_{x_i}\right\|^2_{L_2(Q)} = \\ &= \int_Q u^2(x,t) + u_t^2(x,t) + \sum_{i=1}^n u^2_{x_i}(x,t)dQ, \end{aligned} \tag{5}$$

$$\begin{aligned} ||u||^2_W &= \left\|\int_0^t u(x,\tau)d\tau\right\|^2_E = \int_Q u^2(x,t)dQ + \\ &+ \int_Q \left(\int_0^t u(x,\tau)d\tau\right)^2 + \sum_{i=1}^n \left(\int_0^t u_{x_i}(x,\tau)\,d\tau\right)^2 dQ, \end{aligned} \tag{6}$$

$$\begin{aligned} ||v||^2_{W_+} &= \left\|\int_t^T v(x,\tau)d\tau\right\|^2_E = \int_Q v^2(x,t)dQ + \\ &+ \int_Q \left(\int_t^T v(x,\tau)d\tau\right)^2 + \sum_{i=1}^n \left(\int_t^T v_{x_i}(x,\tau)\,d\tau\right)^2 dQ. \end{aligned} \tag{7}$$

By E^-, E^-_+, W^-, W^-_+ we denote corresponding negative spaces with respect to L_2. On every pair of positive and negative spaces, e.g. E and E^- the corresponding bilinear form $(u,v)_{E^-\times E}$ is defined, which is an extension of the natural bilinear form $(\cdot,\cdot)_{L_2(Q)}$ from $L_2(Q)$ to $E^- \times E$ by continuity.

3. A priory inequalities

Using the following lemma we are able to derive some a priori inequalities:

Lemma 1. Let $F \in L_2(Q)$, $G \in L_2([0,T]\times Q)$, $\sup|G| \leq M$. Then, for an arbitrary constant $c > 0$ the following inequality holds:

$$\int_Q e^{-ct} F(x,t) \int_0^t G(t,x,\tau)F(x,\tau)d\tau dQ \leq M\sqrt{\frac{T}{c}}\int_Q e^{-ct} F^2(x,t)dQ$$

Proof. For an arbitrary $x \in \Omega$, applying Cauchy-Schwarz's inequality, we get

$$\int_0^t G\,(t,x,\tau)F(x,\tau)d\tau = \int_0^t e^{\frac{c\tau}{2}}\,G(t,x,\tau)\cdot e^{-\frac{c\tau}{2}}F(x,t)d\tau \le$$

$$\le \left(\int_0^t e^{c\tau}\,G^2(t,x,\tau)d\tau\right)^{\frac{1}{2}}\cdot\left(\int_0^t e^{-c\tau}\,F^2(x,\tau)d\tau\right)^{\frac{1}{2}} \le M\cdot\left(\int_0^t e^{c\tau}\,d\tau\right)^{\frac{1}{2}}\cdot\left(\int_0^t e^{-c\tau}\,F^2(x,\tau)d\tau\right)^{\frac{1}{2}} =$$

$$= M\cdot\frac{(e^{ct}-1)^{\frac{1}{2}}}{\sqrt{c}}\cdot\left(\int_0^t e^{-c\tau}\,F^2(x,\tau)d\tau\right)^{\frac{1}{2}} \le \frac{M}{\sqrt{c}}\cdot e^{\frac{ct}{2}}\left(\int_0^t e^{-c\tau}\,F^2(x,\tau)d\tau\right)^{\frac{1}{2}}.$$

And further applying Cauchy-Schwarz's inequality again we get

$$\int_Q e^{-ct}\,F(x,t)\int_0^t G\,(t,x,\tau)F(x,\tau)d\tau dQ \le \int_Q e^{-ct}\,F(x,t)\cdot\frac{M}{\sqrt{c}}\cdot e^{ct/2}\cdot\left(\int_0^t e^{-c\tau}\,F(x,\tau)d\tau\right)^{1/2} dQ =$$

$$= \frac{M}{\sqrt{c}}\cdot\int_Q e^{-ct/2}\,F(x,t)\cdot\left(\int_0^t e^{-c\tau}\,F(x,\tau)d\tau\right)^{1/2} dQ \le$$

$$\le \frac{M}{\sqrt{c}}\cdot\left(\int_Q e^{-ct}\,F^2(x,t)dQ\right)^{1/2}\cdot\left(\int_Q\int_0^t e^{-c\tau}\,F^2(x,\tau)dQ\right)^{1/2} \le$$

$$\le \frac{M}{\sqrt{c}}\cdot\left(\int_Q e^{-ct}\,F^2(x,t)dQ\right)^{\frac{1}{2}}\cdot\left(\int_Q e^{-ct}\,F^2(x,t)dQ\right)^{\frac{1}{2}} =$$

$$= M\sqrt{\frac{T}{c}}\cdot\left(\int_Q e^{-ct}\,F^2(x,t)dQ\right)^{1/2}\cdot\left(\int_Q e^{-ct}\,F^2(x,t)dQ\right)^{1/2} = M\sqrt{\frac{T}{c}}\cdot\int_Q e^{-ct}\,F^2(x,t)dQ.$$

Similar result might be established to be used in adjoin problem investigation.

Lemma 2. Let $F \in L_2(Q)$, $G \in L_2([0,T]\times Q)$ and $sup|G| \le M$. Then, for arbitrary negative constant c, the following inequality holds

$$\int_Q e^{-ct}\,F(x,t)\int_t^T G\,(t,x,\tau)F(x,\tau)d\tau dQ \le M\sqrt{\frac{T}{|c|}}\int_Q e^{-ct}\,F^2(x,t)dQ.$$

The next lemmas are the main result of the paper.

Lemma 3. There exists a constant $c_1 > 0$, such that for an arbitrary function $u \in C_{BR}^{\infty}$ the following inequality holds

$$||\mathcal{L}_\varepsilon u||_{E_+^-} \le c_1||u||_E.$$

Proof. For arbitrary $u \in C_{BR}^{\infty}, v \in C_{BR^*}^{\infty}$ we will consider the value of a bilinear form $(\mathcal{L}_\varepsilon u, v)_{L_2(Q)}$. We have:

$$(\mathcal{L}_\varepsilon u, v)_{L_2(Q)} \quad = \left(-\varepsilon\frac{\partial u}{\partial t}, v\right)_{L_2(Q)} + (u,v)_{L_2(Q)} - \sum_{i=1}^n\left(\int_0^t\frac{\partial}{\partial x_i}\left(K_i(t,\tau,x)\frac{\partial u(x,\tau)}{\partial x_i}\right)d\tau, v\right)_{L_2(Q)}.$$

Applying Cauchy-Schwarz's inequality we get

$$\left|(\mathcal{L}_\varepsilon u, v)_{L_2(Q)}\right| \le \left|\left(-\varepsilon\frac{\partial u}{\partial t}, v\right)_{L_2(Q)}\right| + \left|(u,v)_{L_2(Q)}\right| + \sum_{i=1}^n\left|\left(\int_0^t\frac{\partial}{\partial x_i}\left(K_i(t,\tau,x)\frac{\partial u(x,\tau)}{\partial x_i}\right)d\tau, v\right)_{L_2(Q)}\right|$$

$$\le \varepsilon \parallel u_t \parallel_{L_2(Q)}\parallel v \parallel_{L_2(Q)} + \parallel u \parallel_{L_2(Q)}\parallel v \parallel_{L_2(Q)} + \sum_{i=1}^n\left|\left(\int_0^t\frac{\partial}{\partial x_i}\left(K_i(t,\tau,x)\frac{\partial u(x,\tau)}{\partial x_i}\right)d\tau, v\right)_{L_2(Q)}\right|.$$

Applying Gauss formula to the terms of the sum from the latest formula and using integration by parts with the respect to *t* we obtain

$$I=\left|\iint_Q \int_0^t K_i\,(t,\tau,x)u_{x_i}(x,\tau)d\tau v_{x_i}(x,t)dQ\right|=$$
$$=\left|\iint_Q \int_0^t K_i\,(t,\tau,x)u_{x_i}(x,\tau)d\tau \frac{d}{dt}\left(\int_t^T v_{x_i}\,(x,s)ds\right)dQ\right|\le$$
$$\le\left|\iint_Q \int_0^t K_i\,'(t,\tau,x)u_{x_i}(x,\tau)d\tau\int_t^T v_{x_i}\,(x,s)dsdQ\right|+\left|\iint_Q K_i\,(t,t,x)u_{x_i}(x,t)\int_t^T v_{x_i}\,(x,s)dsdQ\right|\le$$
$$\le M\left|\iint_Q \int_0^t \left|u_{x_i}(x,\tau)\right|d\tau\int_t^T \left|v_{x_i}(x,s)\right|dsdQ\right|+M\left|\iint_Q \left|u_{x_i}(x,\tau)\right|\int_t^T \left|v_{x_i}(x,s)\right|dsdQ\right|.$$

Using the Cauchy and Friedrichs' inequalities we get

$$I\le M\left\|\int_0^t\left|u_{x_i}(x,\tau)\right|d\tau\right\|_{L_2(Q)}\cdot\left\|\int_t^T\left|v_{x_i}(x,s)\right|ds\right\|_{L_2(Q)}+M\parallel u_{x_i}\parallel_{L_2(Q)}\cdot\left\|\int_t^T\left|v_{x_i}(x,s)\right|ds\right\|_{L_2(Q)}\le$$
$$\le c_1M\left(\left\|u_{x_i}\right\|_{L_2(Q)}\cdot\left\|v_{x_i}\right\|_{L_2(Q)}+\left\|v_{x_i}\right\|_{L_2(Q)}\cdot\left\|u_{x_i}\right\|_{L_2(Q)}\right)$$

Thus $I\le c_2\parallel u\parallel_E\parallel v\parallel_{E_+}$. Therefore,

$$\left|(\mathcal{L}_\varepsilon u,v)_{L_2(Q)}\right|\le\varepsilon\|u_t'\|_{L_2(Q)}\|v\|_{L_2(Q)}+\|u\|_{L_2(Q)}\|v\|_{L_2(Q)}+c_2\|u\|_E\|v\|_{E_+}\le c_4\|u\|_E\|v\|_{E_+}.$$

Next, we can divide by $\parallel v\parallel_{E_+}$ and taking the supremum over all $y\in C_{BR}^\infty$ will finish the proof. □

In the similar way the following lemma can be proven.

Lemma 4. There exists a constant $c_1>0$, such that for an arbitrary function $v\in C_{BR}^\infty$ the following inequality holds:

$$\|\mathcal{L}_\varepsilon^*v\|_{E^-}\le c_1\|v\|_{E_+}.$$

The inequalities stated in lemmas 2 and 3 allow us to extend the operators $\mathcal{L}_\varepsilon$ and $\mathcal{L}_\varepsilon^*$ from their domains to the spaces E and E_+, respectively, by continuity. We will keep the same notation for the extended operators. We note that the inequalities stated in Lemmas 3 and 4 will hold now for all $u\in E$, $v\in E_+$.

Lemma 5. There exists a constant $c_0>0$, such that for an arbitrary function $v\in E_+$ the following inequality holds:

$$\|\mathcal{L}_\varepsilon^*v\|_{E^-}\ge c_0\|v\|_{W_+}.$$

Proof. First, let $u\in C_{BR}^\infty$. Following [24] let us consider the function u, that is defined by

$$\int_t^T v\,(x,s)ds=e^{-ct}u(x,t),$$

where the value of the constant $c>0$ we will define later. The function u, defined in this way belongs to the space C_{BR}^∞. It is clear, that $v(x,t)=-\left(e^{-ct}u(x,t)\right)_t'$.

Consider the corresponding value of the bilinear form $(\mathcal{L}_\varepsilon u,v)=(-\varepsilon u_t,v)+(\mathcal{L}u,v)$, where operator $\mathcal{L}$ is given by

$$\mathcal{L}u=u-\int_0^t\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(K_i(t,\tau,x)\frac{\partial u(x,\tau)}{\partial x_i}d\tau\right).$$

Let us denote $(\mathcal{L}u,v)_{L_2(Q)}=I_1+I_2$, where

$$I_1=(u,v)_{L_2(Q)}=-\int_Q u\,(x,t)\left(e^{-ct}u(x,t)\right)'_t=-\int_Q e^{ct}\cdot e^{-ct}u(x,t)\left(e^{-ct}u(x,t)\right)'_t dQ=$$
$$=-\frac{1}{2}\int_\Omega\int_0^T e^{ct}\frac{d}{dt}\left(e^{-ct}u(x,t)\right)^2dtd\Omega.$$

Integrating by parts with the respect to the variable t, we get

$$I_1=-\frac{1}{2}\int_\Omega\left(e^{ct}\left(e^{-ct}u(x,t)\right)^2|_{t=0}^{t=T}-\int_0^T ce^{ct}\cdot e^{-2ct}u^2(x,t)dt\right)d\Omega=$$
$$=\frac{1}{2}\int_\Omega\left(u^2(x,0)+\int_0^T ce^{ct}\cdot e^{-2ct}u^2(x,t)dt\right)d\Omega\ge\frac{c}{2}\int_Q e^{-ct}\,u^2(x,t)dQ=\frac{c}{2}\parallel e^{-ct/2}u\parallel_{L_2(Q)}.$$

Next, consider the second term

$$I_2 = -\int_Q \int_0^t \sum_{i=1}^n \left(K_i(t,\tau,x)u_{x_i}(x,\tau)\right)_{x_i} d\tau v(x,t)dQ.$$

Using the Gauss theorem, we get

$$I_2 \quad = -\int_Q \int_0^t \sum_{i=1}^n \left(K_i(t,\tau,x)u_{x_i}(x,\tau)\right)_{x_i} d\tau v(x,t)dQ = \int_Q \int_0^t \sum_{i=1}^n K_i\,(t,\tau,x)u_{x_i}(x,\tau)d\tau v_{x_i}(x,t)dQ =$$

$$= \int_\Omega \int_0^T \int_0^t \sum_{i=1}^n K_i\,(t,\tau,x)u_{x_i}(x,t)d\tau v_{x_i}(x,t)dtd\Omega.$$

Now, integrating by parts and using definition of u we derive

$$\int_0^T \int_0^t K_i\,(t,\tau,x)u_{x_i}(x,\tau)d\tau \cdot v_{x_i}(x,t)dt =$$

$$= \int_0^t K_i\,(t,\tau,x)u_{x_i}(x,t)d\tau \cdot \int_t^T - v_{x_i}(x,s)ds|_{t=0}^{t=T}$$

$$-\int_0^T \left(\int_0^t K_i\,(t,\tau,x)u_{x_i}(x,\tau)d\tau\right)_t' \cdot \int_t^T - v_{x_i}(x,s)dsdt =$$

$$= \int_0^T \left(\int_0^t K_i\,(t,\tau,x)u_{x_i}(x,\tau)d\tau\right)_t' \cdot \int_t^T v_{x_i}\,(x,s)dsdt =$$

$$= \int_0^T \left(\int_0^t K_i\,(t,\tau,x)u_{x_i}(x,\tau)d\tau\right)_t' \cdot e^{-ct}u_{x_i}(x,t)dt =$$

$$= \int_0^T \left(\int_0^t K_i'\,(t,\tau,x)u_{x_i}(x,\tau)d\tau + K_i(t,t,x)u_{x_i}(x,t)\right) \cdot e^{-ct}u_{x_i}(x,t)dt.$$

Hence

$$I_2 = \sum_{i=1}^n \int_Q \left(\int_0^t K_i'\,(t,\tau,x)u_{x_i}(x,\tau)d\tau + K_i(t,t,x)u_{x_i}(x,t)\right) \cdot e^{-ct}u_{x_i}(x,t)dQ =$$

$$= \sum_{i=1}^n \int_Q \int_0^t K_i'\,(t,\tau,x)u_{x_i}(x,\tau)d\tau e^{-ct}u_{x_i}(x,t)dQ + \sum_{i=1}^n \int_Q K_i\,(t,t,x)e^{-ct}u_{x_i}^2(x,t)dQ.$$

Now, applying lemma 1, we get

$$\int_Q e^{-ct}\,u_{x_i}(x,t) \int_0^t K_i'\,(t,\tau,x)u_{x_i}(x,\tau)d\tau dQ \le M\sqrt{\frac{T}{c}}\int_Q e^{-ct}\,u_{x_i}^2(x,t)dQ.$$

Using this in the latest equality, we obtain

$$I_2 \ge \sum_{i=1}^n \int_Q K_i\,(t,t,x)e^{-ct}u_{x_i}^2(x,t)dQ - \sum_{i=1}^n \left|\int_Q \int_0^t K_i'\,(t,\tau,x)u_{x_i}(x,\tau)d\tau e^{-ct}u_{x_i}(x,t)dQ\right| \ge$$

$$\ge \sum_{i=1}^n \alpha \int_Q e^{-ct}\,u_{x_i}^2(x,t)dQ - \sum_{i=1}^n M\sqrt{\frac{T}{c}}\int_Q e^{-ct}\,u_{x_i}^2(x,t)dQ = \sum_{i=1}^n \left(\alpha - M\sqrt{\frac{T}{c}}\right) \| \,e^{-ct/2}u_{x_i}\,\|_{L_2(Q)}^2.$$

Summing up inequalities for I_1 and I_2, it follows that

$$(\mathcal{L}u,v)_{L_2(Q)} = I_1 + I_2 \ge \frac{c}{2} \|\, e^{-ct/2}u\,\|_{L_2(Q)}^2 + \left(\alpha - M\sqrt{\frac{T}{c}}\right)\sum_{i=1}^n \|\, e^{-ct/2}u_{x_i}\,\|_{L_2(Q)}^2.$$

Considering $(-\varepsilon u_t, v)_{L_2(Q)}$, using Cauchy inequality, one can establish the following inequality

$$(-\varepsilon u_t, v)_{L_2(Q)} \ge \varepsilon \|\, e^{-ct/2}u_t\,\|_{L_2(Q)}^2 - c\varepsilon \|\, e^{-ct/2}u_t\,\|_{L_2(Q)} \|\, e^{-ct/2}u\,\|_{L_2(Q)}.$$

Putting all together and using obvious estimate $\left\|e^{-ct/2}u\right\|_{L_2(Q)} \geq c_0\|u\|_{L_2(Q)}$, we finally get

$$(\mathcal{L}_\varepsilon u, v)_{L_2(Q)} \geq \frac{\varepsilon}{2} \parallel e^{-\frac{ct}{2}}u_t \parallel^2_{L_2(Q)} - c\varepsilon \parallel e^{-\frac{ct}{2}}u_t \parallel_{L_2(Q)} \parallel e^{-\frac{ct}{2}}u \parallel_{L_2(Q)} +$$
$$+\frac{c}{4} \parallel e^{-\frac{ct}{2}}u \parallel^2_{L_2(Q)} + c_1 \parallel u \parallel^2_E,$$

for some c_1. Recalling (4) we claim that $c > 0$ can be chosen in a way such that

$$\varepsilon \leq \frac{1}{2c} \leq \frac{\alpha^2}{2TM^2}.$$

In this case $\alpha - M\sqrt{T/c} > 0$ and moreover, Cauchy inequality and $2\varepsilon c < 1$ give

$$\frac{\varepsilon}{2} \parallel e^{-\frac{ct}{2}}u_t \parallel^2_{L_2(Q)} - c\varepsilon \parallel e^{-\frac{ct}{2}}u_t \parallel_{L_2(Q)} \parallel e^{-\frac{ct}{2}}u \parallel_{L_2(Q)} + \frac{c}{4} \parallel e^{-\frac{ct}{2}}u \parallel^2_{L_2(Q)} \geq 0.$$

Thus one can derive

$$(\mathcal{L}_\varepsilon u, v)_{L_2(Q)} \geq c_1 \parallel u \parallel^2_E.$$

We outline that

$$||v||^2_{W_+} = \left\|\int_t^T v(x,\tau)d\tau\right\|^2_E = \|e^{-ct}u\|^2_E = \int_Q e^{-2ct}u^2 + ((e^{-ct}u)')^2 + \sum_{i=1}^n e^{-2ct}u^2_{x_i}\,dQ =$$

$$= \int_Q e^{-2ct}\left[u^2 + c^2u^2 - 2cuu_t + u_t^2 + \sum_{i=1}^n u^2_{x_i}\right]dQ \leq \int_Q e^{-2ct}\left[u^2 + 2c^2u^2 + 2u_t^2 + \sum_{i=1}^n u^2_{x_i}\right]dQ$$
$$\leq c_2^{-2} \parallel u \parallel^2_E,$$

where c_2^{-2} is some constant. Finally, applying the Schwarz inequality to the left-hand side, we have

$$\parallel u \parallel_E \parallel \mathcal{L}^*_\varepsilon v \parallel_{E^-} \geq (u, \mathcal{L}^*_\varepsilon v)_{L_2(Q)} \geq c_1 \parallel u \parallel^2_E \geq c_1c_2 \parallel u \parallel_E \parallel v \parallel_{W_+}.$$

and $\parallel \mathcal{L}^*_\varepsilon v \parallel_{E^-} \geq c_3 \parallel v \parallel_{W_+}$, for an arbitrary $u \in C^\infty_{BR}$. For the remaining functions in E the desired result can be established by using the density argument. Namely, let $v \in E$ (not necessarily smooth). Due to the density of embedding $C^\infty_{BR} \subset E_+$, the function v can be approximated by a sequence of functions $v_m \in C^\infty_{BR}$. That is,

$$||v - v_m||_{E_+} \to 0 \text{ as } m \to \infty.$$

From Lemma 3 it follows that $||\mathcal{L}^*_\varepsilon v - \mathcal{L}^*_\varepsilon v_m||_{E^-} \leq c_1||v - v_m||_{E_+}$. Thus, $\mathcal{L}^*_\varepsilon v_m \to \mathcal{L}^*_\varepsilon v$ in the space E^-. For each element v_m, the inequality $C||v_m||_{W_+} \leq ||\mathcal{L}^*_\varepsilon v_m||_{E^-}$ holds as proved above. It remains to note that due to the convergence in the space E, we have convergence in W_+, i.e., $||v - v_m||_{W_+} \to 0$. Finally, passing to the limit as $m \to \infty$ in the last inequality, we obtain the desired result. □

Lemma 6. There exists some constant $c_0 > 0$, such that for an arbitrary function $u \in E$ the following inequality holds:

$$\|\mathcal{L}_\varepsilon u\|_{E^-_+} \geq c_0\|u\|_W.$$

Proof. In a similar way to the proof of the lemma 5 let $v \in C^\infty_{BR^*}$ and we consider an auxiliary function u, defined as follows

$$\int_0^t u(x,s)ds = e^{-ct}v(x,t),$$

where the value of the constant $c < 0$ will be defined later. Note, that function u, defined in this way belongs to the space C^*_{BR}. It is clear, that $u(x,t) = \left(e^{-ct}v(x,t)\right)'_t$. As above we consider the value of the bilinear form

$$(\mathcal{L}_\varepsilon u, v) = (u, \mathcal{L}^*_\varepsilon v) = (\varepsilon v_t, u) + I_1 + I_2.$$

By virtue of Gauss theorem, Cauchy and Cauchy-Schwarz's inequality, and integration by parts it could be prove that

$$I_1 = (u,v)_{L_2(Q)} = \int_Q v(x,t)\left(e^{-ct}v(x,t)\right)'_t =$$

$$= \int_Q e^{ct} \cdot e^{-ct}v(x,t)\left(e^{-ct}v(x,t)\right)'_t dQ = \frac{1}{2}\int_\Omega \int_0^T e^{ct}\frac{d}{dt}\left(e^{-ct}v(x,t)\right)^2 dt d\Omega.$$

And thus

$$I_1 = \frac{1}{2}\int_\Omega \left(e^{ct}\left(e^{-ct}v(x,t)\right)^2 |_{t=0}^{t=T} - \int_0^T c\, e^{ct} \cdot e^{-2ct} v^2(x,t)dt \right) d\Omega \geq$$
$$\geq -\frac{c}{2}\int_Q e^{-ct}\, v^2(x,t)dQ = -\frac{c}{2} \parallel e^{-\frac{ct}{2}} v \parallel_{L_2(Q)}.$$

Using transformations

$$\int_0^T \int_t^T K_i\,(\tau,t,x) v_{x_i}(x,\tau) d\tau u_{x_i}(x,t)dt =$$
$$= \int_t^T K_i\,(\tau,t,x) v_{x_i}(x,\tau)d\tau \cdot \int_0^t u_{x_i}\,(x,s)ds|_{t=0}^{t=T}$$
$$-\int_0^T \left(\int_t^T K_i\,(\tau,t,x) v_{x_i}(x,\tau)d\tau \right)_t' \cdot \int_0^t u_{x_i}\,(x,s)dsdt =$$
$$= \int_0^T \left(\int_t^T K_i\,(\tau,t,x) v_{x_i}(x,\tau)d\tau \right)_t' \cdot \int_0^t u_{x_i}\,(x,s)dsdt =$$
$$= \int_0^T \left(\int_t^T K_i\,(\tau,t,x) v_{x_i}(x,\tau)d\tau \right)_t' \cdot e^{-ct} v_{x_i}(x,t)dt =$$
$$= \int_0^T \left(\int_t^T K_i{}'(\tau,t,x) v_{x_i}(x,\tau)d\tau - K_i(t,t,x)v_{x_i}(x,t) \right) \cdot e^{-ct} v_{x_i}(x,t)dt,$$

We are able to conclude that

$$I_2 = -\int_Q \int_t^T \sum_{i=1}^n \left(K_i(\tau,t,x) v_{x_i}(x,\tau) \right)_{x_i} d\tau \cdot u(x,t)dQ =$$
$$= \int_Q \int_t^T \sum_{i=1}^n K_i\,(\tau,t,x) v_{x_i}(x,\tau)d\tau \cdot u_{x_i}(x,\tau)dQ =$$
$$= \int_\Omega \int_0^T \int_t^T \sum_{i=1}^n K_i\,(\tau,t,x) v_{x_i}(x,\tau)d\tau \cdot u_{x_i}(x,t)dtd\Omega =$$
$$= -\sum_{i=1}^n \int_Q \left(\int_0^t K_i'(\tau,t,x) v_{x_i}(x,\tau)d\tau + K_i(t,t,x)v_{x_i}(x,t) \right) \cdot e^{-ct} v_{x_i}(x,t)dQ =$$
$$= \sum_{i=1}^n \int_Q \int_0^t K_i'(\tau,t,x) v_{x_i}(x,\tau)d\tau\, e^{-ct} \cdot v_{x_i}(x,t)dQ + \sum_{i=1}^n \int_Q K_i\,(t,t,x) e^{-ct} v_{x_i}^2(x,t)dQ.$$

Utilizing lemma 2 we obtain

$$I_2 \geq \sum_{i=1}^n \int_Q K_i\,(t,t,x) e^{-ct} v_{x_i}^2 dQ - \sum_{i=1}^n \left| \int_Q \int_0^t K_i{}'(\tau,t,x) u_{x_i}(x,\tau)d\tau\, e^{-ct} v_{x_i}(x,t)dQ \right| \geq$$
$$\geq \sum_{i=1}^n \alpha \int_Q e^{-ct}\, v_{x_i}^2(x,t)dQ - \sum_{i=1}^n M \sqrt{\frac{T}{|c|}} \int_Q e^{-ct}\, v_{x_i}^2 dQ = \left(\alpha - M\sqrt{\frac{T}{|c|}} \right) \sum_{i=1}^n \parallel e^{-ct/2} v_{x_i} \parallel_{L_2(Q)}^2.$$

Summing everything up we further get

$$(\mathcal{L}u,v)_{L_2(Q)} = I_1 + I_2 \geq -\frac{c}{2} \parallel e^{-\frac{ct}{2}} v \parallel_{L_2(Q)}^2 + \left(\alpha - M\sqrt{\frac{T}{|c|}} \right) \sum_{i=1}^n \parallel e^{-\frac{ct}{2}} v_{x_i} \parallel_{L_2(Q)}^2.$$

We now estimate $(\varepsilon v_t, u)_{L_2(Q)}$ as follows:

$$(\varepsilon v_t, u)_{L_2(Q)} = \varepsilon \int_Q v_t\,(x,t) \left(e^{-ct} v(x,t) \right)'_t dQ = \varepsilon \int_Q v_t\,(x,t) \left(-ce^{-ct} v(x,t) \right) + e^{-ct} v_t(x,t))dQ =$$
$$= \varepsilon \parallel e^{-ct/2} v_t \parallel_{L_2(Q)}^2 - c\varepsilon \int_Q e^{-ct}\, v_t(x,t)v(x,t)dQ.$$

Now, applying the Cauchy inequality to the last term we can get

$$(\varepsilon v_t, u)_{L_2(Q)} \geq \varepsilon \parallel e^{-\frac{ct}{2}} v_t \parallel^2_{L_2(Q)} - \left| c\varepsilon \int_Q e^{-ct} v_t(x,t) v(x,t) dQ \right| \geq$$
$$\geq \varepsilon \parallel e^{-ct/2} v_t \parallel^2_{L_2(Q)} + c\varepsilon \parallel e^{-ct/2} v_t \parallel_{L_2(Q)} \parallel e^{-ct/2} v \parallel_{L_2(Q)}.$$

Summing up it with estimations for I_1 and I_2 we obtain

$$(\mathcal{L}^* v, u)_{L_2(Q)} \geq \varepsilon \parallel e^{-ct/2} v_t \parallel^2_{L_2(Q)} + c\varepsilon \parallel e^{-ct/2} v_t \parallel^2_{L_2(Q)} \parallel e^{-ct/2} v \parallel^2_{L_2(Q)} -$$
$$- \frac{c}{2} \parallel e^{-ct/2} v \parallel^2_{L_2(Q)} + \left(\alpha - M \sqrt{\frac{T}{|c|}} \right) \sum_{i=1}^{n} \parallel e^{-ct/2} v_{x_i} \parallel^2_{L_2(Q)} \geq$$
$$\geq \frac{\varepsilon}{2} \parallel e^{-ct/2} v_t \parallel^2_{L_2(Q)} + c\varepsilon \parallel e^{-ct/2} v_t \parallel^2_{L_2(Q)} \parallel e^{-ct/2} v \parallel^2_{L_2(Q)} -$$
$$- \frac{c}{4} \parallel e^{-ct/2} v \parallel^2_{L_2(Q)} + c_1 \parallel e^{-ct/2} v \parallel^2_{E},$$

where $c_1 < \min\left\{ \frac{\varepsilon}{2}, \frac{|c|}{4}, \alpha - M\sqrt{\frac{T}{|c|}} \right\}$. Recalling (4) we claim that for some $c \leq 0$ such that

$$\varepsilon \leq \frac{1}{2|c|} < \frac{\alpha^2}{2TM^2}$$

we will get

$$\frac{\varepsilon}{2} \parallel e^{-\frac{ct}{2}} v_t \parallel^2_{L_2(Q)} + \frac{|c|}{4} \parallel e^{-\frac{ct}{2}} v \parallel^2_{L_2(Q)} \geq |c|\varepsilon \parallel e^{-\frac{ct}{2}} v_t \parallel_{L_2(Q)} \parallel e^{-\frac{ct}{2}} v \parallel_{L_2(Q)}.$$

And therefore

$$(\mathcal{L}^*_\varepsilon v, u)_{L_2(Q)} \geq c_2 \parallel v \parallel^2_{E_+} \geq c_3 \parallel v \parallel_{E_+} \parallel u \parallel_W.$$

Finally,

$$\parallel \mathcal{L}_\varepsilon u \parallel_{E^-} \parallel v \parallel_{E_+} \geq (\mathcal{L}_\varepsilon v, u)_{L_2(Q)} \geq c_2 \parallel v \parallel^2_{E_+} = c_2 \parallel v \parallel_{E_+} \parallel u \parallel_W,$$

which lead us to the result

$$\parallel \mathcal{L}_\varepsilon u \parallel_{E^-_+} \geq c_2 \parallel u \parallel_W,$$

for smooth u. The rest of the proof is similar to the lemma 5.

As a summary we conclude, that a following theorem for operators $\mathcal{L}_\varepsilon, \mathcal{L}^*_\varepsilon$ holds:

Theorem 1. There exist constants $c_0, c_1 > 0$, such that for any $u \in E, v \in E_+$ the following inequalities hold

$$\begin{cases} c_0 ||u||_W \leq ||\mathcal{L}_\varepsilon u||_{E^-_+} \leq c_1 ||u||_E, \\ c_0 ||v||_{W_+} \leq ||\mathcal{L}^*_\varepsilon v||_{E^-} \leq c_1 ||v||_{E_+}. \end{cases} \quad (8)$$

4. Generalized solvability

Consider a problem

$$\mathcal{L}_\varepsilon u = f, f \in E^-_+. \quad (9)$$

We will interpret its solutions in the following senses:

Definition 1. The function $u(x,t) \in E$ is called the solution of the problem (9) with the right-hand side $f \in E^-_+$ if there exists a sequence of functions $u_i(x,t) \in C^\infty_{BR}$, for which the following property holds:

$$||u - u_i||_E \to 0, ||\mathcal{L}u_i - f||_{E^-_+} \to 0, i \to \infty. \quad (10)$$

Definition 2. The function $u(x,t) \in E$ is called the weak solution of the problem (9) with the right hand side $f \in E^-_+$, if the equation

$$(\mathcal{L}_\varepsilon u, v)_{E^-_+, E_+} = (f, v)_{W^-_+, W_+} \quad (11)$$

holds for any functions $v \in C^\infty_{BR^*}$.

The solutions of the adjoin problem are defined in a similar way.

Based on the estimations (8) and on the work [7], we state the theorems of generalized solvability.

Theorem 2. Definitions 1 and 2 are equivalent.

Theorem 3. For an arbitrary element $f \in W^-_+$ there exists a unique solution of (9) in the sense of the definitions 1 and 2.

Theorem 4. Let $u(x,t)$ be the solution of the problem (9), with the right-hand side $f \in W_+^-$ in the sense of the definitions 1 and 2. Then, the estimation $||u||_E \le c||f||_{W_+^-}$ holds, where the constant c does not depend on f.

Similar theorems can be stated for the adjoin problem.

Remark. Using the proven a priori inequalities, it is also possible to prove the results on existence of optimal control, to establish certain differential properties of the performance criterion, properties of the regularized problem, construct and prove the convergence of numerical methods for finding generalized solutions and optimal controls, etc.

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Adaptive Method Approximation of Experimental Data

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Abstract

Adoptive algorithm for signal approximation based on gradient approach described. The main goal of the method is to define parameter's vector used for data approximation. For this purpose, two types of residuals investigated. Iterative procedure's convergence analysis conducted using Lyapunov methods. Efficiency of the described algorithm experimentally proved by detection of chemical components in plants.

Keywords [3]

adaptation, real-time signal, approximation, basic function, spectral data, differencing schemes.

1. Introduction

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Despite the fact that there are many works devoted to the problems of approximation both continuous and discrete signals, important practical problems arise, especially in the field of Informatics and applied mathematics [1]. It requires development and testing of new approaches of experimental data approximation. First, it comes from the fact that data processing is mostly conducted in real time. Besides, algorithms should meet strict conditions: they should be constructive, focused on optimal performance and real-time problem solving [6, 7].

These requirements, in our view, meet the below described adaptive algorithms based on gradient approach. Proposed approaches described for the approximation of the continuous processes only, but it is easy to derive discrete counterparts on their basis. As a rule, signals are measured into discrete moments. That is why differencing schemes could prove to be more effective for use.

The problem of analysis of convergence of the following iterative procedures arise when using proposed algorithms. It can be done on the basis of Lyapunov methods [2], practical resistance and special criteria of stability [3].

Suggested methods could be effectively applied for the detection of chemical and biological analysis of the spectral data. Model examples for the approximation of the continuous signals are given to demonstrate their effectiveness.

2. Approximation of continuous signals

Let suppose that we know of uninterrupted, for ease of scalar signal $x = \phi(t),\ t_0 \le t \le T$, which needs to be approximate with parametrically given assemblage

$$x(t) \approx \ \psi(t,\alpha) = \psi(t,\alpha_1,\alpha_2,\dots\alpha_n). \quad (1)$$

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If the signal is defined on $[t_0, t]$, the task of parameters α vector adaptive correction is to minimize certain residual [5]. For this purpose, we will consider two types of residuals:

a) directly at the moment t

$$I_1(\alpha) = \left(\psi(t,\alpha) - \phi(t)\right)^2; \quad (2)$$

b) mean square approximation of $[t_0, t]$

$$I_2(\alpha) = \int_{t_0}^{t} \left(\psi(\tau,\alpha) - \phi(\tau)\right)^2 d\tau. \quad (3)$$

For correction of parameters in order to minimize residual (2) uninterrupted iterative procedure is written down

$$\frac{d\alpha}{dt} = -grad_\alpha I_1(\alpha) = -2\left(\psi(t,\alpha) - \phi(t)\right) grad_\alpha \psi(t,\alpha) \quad (4)$$

with some initial data

$$\alpha(t_0) = \alpha^{(0)}. \quad (5)$$

To find vector parameters α solution of Cauchy problem is needed (4), (5). If there is a stationary problem solution (4), (5), that is $\alpha(t) \to \bar{\alpha}, \quad t \to \infty$, you can be taken as a solution of given problem. It is necessary to notice that for solving some practical tasks such a simple procedure gives good results.

For the integral residual we write down the same system of ordinary differential equations

$$\frac{d\alpha}{dt} = -grad_\alpha I_2(\alpha) = -2 \int_{t_0}^{t} \left(\psi(\tau,\alpha) - \phi(\tau)\right) grad_\alpha \psi(\tau,\alpha) d\tau. \quad (6)$$

Because in the right part of the system (6) is one of the integral, it can be rewritten in a more constructive view

$$\frac{d^2\alpha}{dt^2} = -2\left(\psi(t,\alpha) - \phi(t)\right) grad_\alpha \psi(t,\alpha). \quad (7)$$

As (7) is a system of ordinary differential equations of order $2n$, recorded in normal form, initial conditions for it must be selected as following

$$\alpha(t_0) = \alpha^{(0)}, \quad \frac{d\alpha(t_0)}{dt} = 0. \quad (8)$$

That means, in another case is needed to solve the problem numerically with one of the methods, such as Runge-Kutta, a Cauchy (7), (8). As in the previous case, if there is a limit $\alpha(t) \to \bar{\alpha}, \quad t \to \infty$, when solving (7), (8), it can be taken as a solution of the given problem.

Comment 1. The original data is to be chosen from the convergence of the proposed iterative procedures.

Convergence of iterative procedures can be investigated on the basis of Lyapunov second method. Therefore, parameter $\alpha^{(0)}$ we will choose from the range of asymptotic stability of appropriate system of ordinary differential equations. If, however there will be performed conditions of the Barbashin-Krasovsky theorem about stability in general, the convergence of iterative procedures will be recorded for any of the original data $\alpha^{(0)} \in E^n$.

Consider a more specific problem of formulated type. Suppose we have a system of basic functions

$$\phi_1(t), \phi_2(t), \ldots, \phi_n(t), \quad t \geq t_0, \quad (9)$$

and function $\psi(t,\alpha)$ will choose as a linear combination of

$$\psi(t,\alpha) = \sum_{j=1}^{n} \alpha_j \phi_j(t). \quad (10)$$

In this case, the system of ordinary differential equations (4) we write in this form

$$\frac{d\alpha_i}{dt} = -2\left(\sum_{j=1}^{n} \alpha_j \phi_j(t) - \phi(t)\right)\phi_i(t), i = 1,2,\ldots,n. \quad (11)$$

System (11) is a linear nonparallel sentence system of ordinary differential equations

$$\frac{d\alpha_i}{dt} = -2\phi_i(t) \sum_{j=1}^{n} \phi_j(t)\alpha_j + 2\phi(t)\phi_i(t), i = 1,2,\ldots,n, \quad (12)$$

that can be re rewritten in a vector-matric form

$$\frac{d\alpha}{dt} = A(t)\alpha + f(t), \quad t \geq t_0. \tag{13}$$

Where

$$\alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_n), f^T(t) = 2\phi(t)\big(\phi_1(t), \phi_2(t), \ldots, \phi_n(t)\big),$$

$$A(t) = \begin{pmatrix} -2\phi_i^2(t) & -2\phi_1(t)\phi_2(t) & \ldots & -2\phi_1(t)\phi_n(t) \\ -2\phi_1(t)\phi_2(t) & -2\phi_2^2(t) & \ldots & -2\phi_2(t)\phi_n(t) \\ \ldots & \ldots & \ldots & \ldots \\ -2\phi_1(t)\phi_n(t) & -2\phi_2(t)\phi_n(t) & \ldots & -2\phi_n^2(t) \end{pmatrix}$$

– symmetric matrix of dimension $n \times n$, T – sign of transposition.

According to Cauchy's formula solution of problem (5), (13) can be written as follows

$$\alpha(t) = W(t, t_0)\alpha^{(0)} + \int_{t_0}^{t} W(t, \tau) f(t) d\tau, \tag{14}$$

where $W(t, \tau)$ – scaled under the moment τ fundamental matrix of a homogeneous system, which corresponds to (13), i.e.

$$\frac{dW}{dt} = A(t)W, \quad W(\tau, \tau) = E_n. \tag{15}$$

By analogy, you can extract systems of differential equations for the integral residual (3) provided (10). In this case, the system (6) could be written in this form

$$\frac{d\alpha_i}{dt} = -2\int_{t_0}^{t} \left(\sum_{j=1}^{n} \alpha_j\, \phi_j(\tau) - \phi(\tau)\right) \phi_i(\tau) d\tau, \; i = 1,2, \ldots, n. \tag{16}$$

System (16) is a linear system of integral-differential equations, which by derivation could be minimized to the system of differential equations of the following form (7)

$$\frac{d^2\alpha_i}{dt^2} = -2\left(\sum_{j=1}^{n} \alpha_j \phi_j\,(t) - \phi(t)\right) \phi_i(t), \; i = 1,2, \ldots, n. \tag{17}$$

A linear system (17) could be written in a vector-matrix form

$$\frac{d\bar{\alpha}}{dt} = \overline{A}(t)\bar{\alpha} + 2\overline{f}, \tag{18}$$

where $\bar{\alpha}^T = \left(\alpha^T, \frac{d\alpha^T}{dt}\right)$, $\bar{f}^T(t) = \left(0^T, f^T(t)\right)$ are vectors of dimension *2n*, $\bar{A}(t) = \begin{pmatrix} 0 & E_n \\ A(t) & 0 \end{pmatrix}$ is a matrix of dimension $2n \times 2n$ with the known elements..

However, a diverse linear system of ordinary differential equations (18), in effect (8), you need to consider under partially fixed initial conditions. In order to find an overall system solution (18) under any$\alpha^{(0)}$, will use the Cauchy formula

$$\bar{\alpha}(t) = \bar{W}(t, t_0)\bar{\alpha}(t_0) + \int_{t_0}^{t} \bar{W}(t, \tau)\bar{f}(\tau) d\tau \tag{19}$$

where$\bar{W}(t, \tau)$ – converging for a moment τ fundamental matrix for homogeneous system

$$\frac{d\bar{\alpha}}{dt} = \bar{A}(t)\bar{\alpha} \,. \tag{20}$$

This matrix satisfies the system (20) with a single initial conditions, i.e.,

$$\frac{d\bar{W}}{dt} = \bar{A}(t)\bar{W}, \quad \bar{W}(t_0, t_0) = E_{2n}. \tag{21}$$

We will copy the formula (20) taking into account structure of the vectors $\bar{\alpha}$, $\bar{f}$ and represent the matrix $\bar{W}$ in modular structural form

$$\bar{W}(t, \tau) = \begin{pmatrix} W^{(1,1)}(t, \tau) & W^{(1,2)}(t, \tau) \\ W^{(2,1)}(t, \tau) & W^{(2,2)}(t, \tau) \end{pmatrix},$$

where $W^{(i,j)}(t, \tau)$ – a square matrix of dimension *n* . In this case, we will reach the following vector correlation

$$\alpha(t) = W^{(1.1)}(t, t_0)\alpha^{(0)} + \int_{t_0}^{t} W^{(1,2)}(t, \tau) f(\tau) d\tau, \qquad 22)$$

$$\frac{d\alpha}{dt} = W^{(2,1)}(t, t_0)\alpha^{(0)} + \int_{t_0}^{t} W^{(2,2)}(t, \tau) f(\tau) d\tau \quad . \tag{23}$$

With the formula (22) (23) is evident that for analysis of setting vector parameters we need only the first correlation. It will be necessary for evaluating the field of convergence of iterative procedures (18) under initial approach.

In some cases, to determine, for example, a stationary modes of change of vector$\alpha(t)$, needed to set border conditions for the derivative

$$\frac{d\alpha(T)}{dt} = \alpha^{(1)}. \tag{24}$$

Then, using the terms (24) and ratio (23), you can set the appropriate original data, which provide

$$\alpha^{(0)} = W^{(2,1)^{-1}}(T, t_0)[\alpha^{(1)} - \int_{t_0}^{T} W^{(2,2)}(T, \tau) f(\tau) d\tau] \tag{25}$$

In this case, solution (22) will present in the final form

$$\alpha(t) = W^{(1,1)}(t, t_0) W^{(2,1)^{-1}}(t, t_0)[\alpha^{(1)} - \\ - \int_{t_0}^{T} W^{(2,2)}(T, \tau) f(\tau) d\tau] + \int_{t_0}^{t} W^{(2,2)}(T, \tau) f(\tau) d\tau \tag{26}$$

In this case, solution (22) will present in the final form

$$\alpha(t) = W^{(1,1)}(t, t_0) W^{(2,1)^{-1}}(t, t_0)[\alpha^{(1)} - \\ - \int_{t_0}^{T} W^{(2,2)}(T, \tau) f(\tau) d\tau] + \int_{t_0}^{t} W^{(2,2)}(T, \tau) f(\tau) d\tau\ . \tag{26}$$

3. Analysis of iterative procedures convergence

Let us conduct analysis of the convergence of iterative procedures using Lyapunov methods of practical stability [4].

Consider two types of residuals (2) and (3).

Case 1. Let us look at the iterative scheme based on minimization of residual (2). Suppose that solution of the Cauchy problem (5), (13) meets the condition $\alpha^{(1)}(t, t_0, \alpha^{(0)}) \to \bar{\alpha} = const, \quad t \to \infty$. Then with the substitute

$$\alpha = \alpha^{(1)}(t, t_0, \alpha^{(0)}) + \nu(t) \tag{27}$$

we'll come to a homogeneous system of linear differential equations with respect to the new variable $\nu(t)$

$$\frac{d\nu}{dt} = A(t)\nu\ , \quad t \geq t_0\ . \tag{28}$$

Then, on the assumption that the original data (5) can be perturbed, analysis of convergence of iterative procedure (13) will be equivalent to the research of stability of solution $\nu(t) \equiv 0, \quad t \geq t_0$ of linear homogeneous system (28). Fair's the next theorem.

Theorem 1. For the convergence of iterative scheme (13) at perturbed initial data, i.e.

$$\alpha(t, t_0, \alpha^{(0)} + \nu^{(0)}) \to \bar{\alpha}, \quad t \to \infty \tag{29}$$

is necessary and sufficient, the converging for a moment fundamental matrix $W(t, t_0)$, to meet the conditions

$$W(t, t_0) \to 0, \text{ for } t \to \infty. \tag{30}$$

The proof of the formulated theorem is based on the framework as a replacement (27), recording and analysis of Cauchy problem solution for a homogeneous system (28)

$$\nu(t) = W(t, t_0)\nu^{(0)}. \tag{31}$$

Here $\nu^{(0)}$ – *n*-dimensional vector of initial data for homogeneous system (28), $\bar{\alpha}$ –*n*-dimensional stationary vector, which is the solution of the given problem.

Comment 2. It should be mentioned that under fulfilling the terms of the formulated theorem convergence of iterative procedure will be performed for any initial data, i.e., in general. It results from the fact that the fundamental matrix does not depends on the original data.

4. Detection of chemical components in the plants and calculation experiment

Let's show constructiveness and effectiveness of the proposed approach primarily based on spectral data processing of the plants contaminated with chemical elements. We assume that plants pollution is generated by some chemical elements and its spectral data received. They are considered as basic for recognition of them in new experimental data. Let us designate basic spectral functions

$$\phi_1(t), \phi_2(t), \ldots, \phi_n(t), \quad t_0 \le t \le T \ ,$$

which represent the spectral data of plants pollution by known chemical elements.

Let $\phi(t), t \in [t_0, T]$ the measured spectral function contaminated by an unknown chemical element. The function $\psi(t,\alpha)$ will choose as a linear combination of (10) $\psi(t,\alpha) = \sum_{j=1}^{n} \alpha_j \phi_j(t)$. System of ordinary differential equations for this case will be (11)

$$\frac{d\alpha_i}{dt} = -2\big(\sum_{j=1}^{n} \alpha_j \phi_j(t) - \phi(t)\big)\phi_i(t), \, i = 1,2,\ldots,n.$$

Let write it in vector-matrix form (13)

$$\frac{d\alpha}{dt} = A(t)\alpha + f(t), \quad t \in [t_0, T].$$

In the case of consideration of discrete signals, solution of system (13) could be represented as following

$$\alpha(i+1) = \alpha(i) + \big(A(t_i)\alpha(i) + f(t_i)\big)\Delta t, \quad t \in [t_0, T],$$

where $\Delta t = t_i - t_{i-1}$ – quantization on a time.

Depending on a type of experimental data to improve the detection of signs of chemical contaminants contribution it is needed to hold some mathematical conversions. In Particular:

- for each selected discrete basic function (with the main chemical pollution) a new basis is built, which is the difference of functions of the old basis and explored experimental data;
- over discrete basic functions and experimental data under investigation it is advisable to make a discrete conversion of Fourier and assume them as a new converted basic and experimental data under investigation.

5. Calculating experiment on detection of chemical components in plants

Numerical experiment conducted under the described above adaptive algorithm. Spectral experimental data on a plant specimens were chosen for the basic functions, which were contaminated by the chemical elements $CaCl$ and $K_2Cr_2O_7$. Spectral values of basic functions displayed on Figure 1. For the recognition of new experimental data for the contamination with selected chemical elements, an adaptive algorithm is applied and the result are shown on Figure 2.

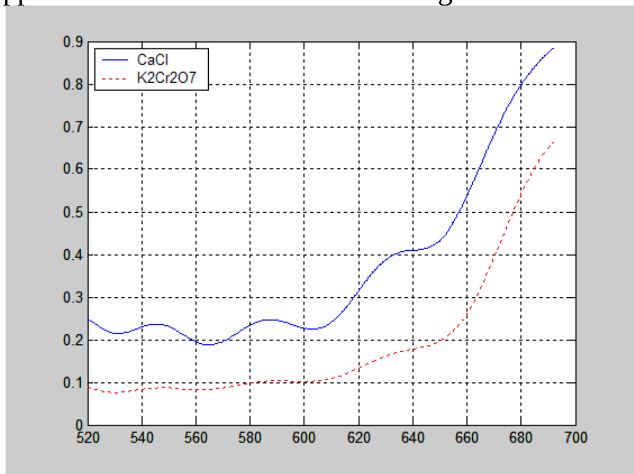

Figure 1. Spectral values of basic functions.

6. Conclusion

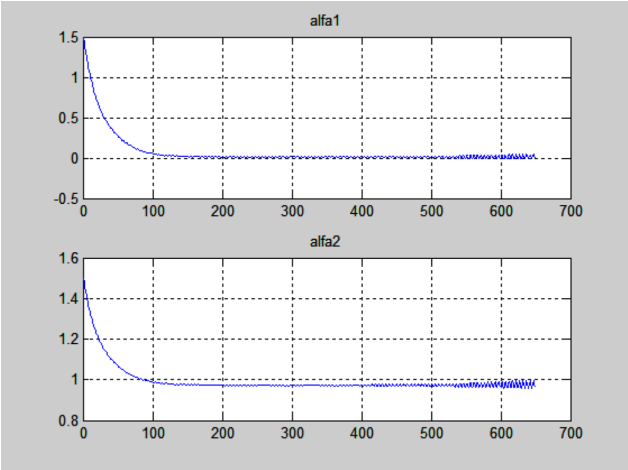

Figure 2. Iterative procedure convergence for experimental data contaminated by the chemical elements $CaCl$ (alfa1) and $K_2Cr_2O_7$ (alfa2).

Unknown parameters vector converges to 0 for the experimental data contaminated by $CaCl$ and to 1 for those contaminated by $K_2Cr_2O_7$ correspondingly. It means that contamination by chemical element $K_2Cr_2O_7$ is recognized

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Regular and complex behaviour in the pendulum system under a magnetic field

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Abstract

Dynamics of two coupled pendulums under a magnetic field are considered by taking into account a dissipation in the system. Inertial components of the pendulums are essentially different, and a ratio of masses is chosen as a small parameter. Padé approximants is used for the magnetic forces approximation. The method of multiple scales is used to construct nonlinear normal modes (NNMs), one of them is a localized one. An appearance of the complex behaviour when the system parameters change is investigated.

Keywords [4]

Paper template, paper formatting, CEUR-WS

1. Introduction

Mathematical and/or physical pendulums are important models presenting typical nonlinear dynamics of different types of nonlinear systems. Series of recent publications represent the theoretical and experimental study of the dynamics of two connected pendulums in a magnetic field [1–3]. Investigation of nonlinear normal vibration modes (NNMs) in such system for a case when masses of these connected pendulums differ significantly is made in [4]. Localized and coupled NNMs are analysed by the small parameter method and numerical simulation. Nonlinear normal modes are an important part in analysis of dynamics of numerous finite-DOF nonlinear systems. The main elements of the NNMs theory and numerous applications are presented, in particular, in reviews [5, 6] and book [7]. Here we analyse localized and coupled NNMs in the pendulum system in the magnetic field taking into account the medium resistance and the damping moment created by the elastic element. That is, the system under study is a two-DOF autonomous and dissipative nonlinear system. We can note also that the problem of localization of oscillations is very important and was investigated in recent decades in numerous publications.

The structure of the article is as follows. Section 2 contains a brief description of the model and the magnetic effect approximation. Sections 3 and 4 describe the study of localized and coupled vibration modes under the influence of magnetic forces and dissipation. Section 5 represents the conclusions obtained from the study of the modes.

2. Basic model

A detailed description of the stand and the system under consideration is given in [1-3]. Here we present only a brief description of the system and the type of interaction of the magnetic forces. Namely, the system is presented in Fig. 1, and the corresponding mathematical model is described by a system of differential equations (1):

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$$\begin{cases} \mu\varepsilon\ddot{\varphi}_1 = \varepsilon\gamma M^*_{mag_1} + \varepsilon M^*_{D1}(\varphi_1,\varphi_2) + \varepsilon\mu M^{(g)^*}(\varphi_1) + M^{(k)^*}(\varphi_1,\varphi_2), \\ \ddot{\varphi}_2 = \varepsilon M^*_{mag_2} + \varepsilon M^*_{D2}(\varphi_1,\varphi_2) + M^{(g)^*}(\varphi_1) + M^{(k)^*}(\varphi_1,\varphi_2), \end{cases} \quad (1)$$

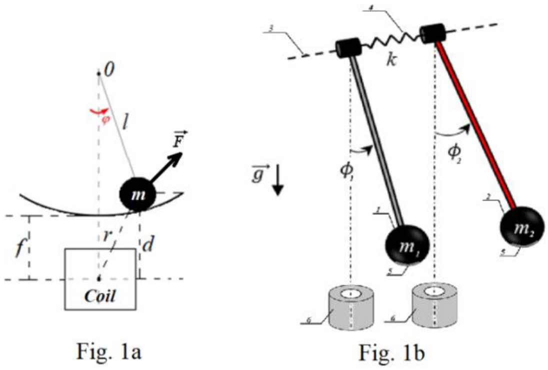
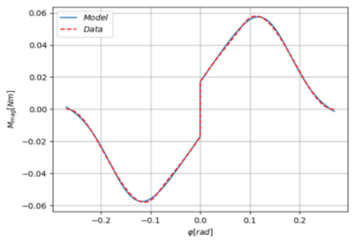


Figure 1. Schemes of the single particle and the magnetic force interaction (Fig.1a) and the coupled pendulums in the stem (Fig.1b).

In the system (1) μ is the ratio of the masses of the pendulums; $M^*_{mag_{1,2}}$ are the characteristics of the magnetic forces, which are reduced to the moment of forces; γ – intensity of magnetic influence; ε is a formal small parameter; $k_l$is the stiffness of the binding elastic element; $M^{(g)^*}, M^{(k)^*}$ are returning moments of gravity and elastic forces, respectively; $M^{(g)^*} = -\frac{r}{I}\sin\varphi_{1,2}$, $M^{(k)^*} = -\frac{k_l}{I}(\varphi_1 - \varphi_2), r = mgs$; we use the shortened Taylor series, $\sin(\varphi) \cong \varphi - \frac{1}{6}\cdot\varphi^3$; $M^*_{D1,2}$ is the damping moments, $M^*_{D1} = -\frac{C_1}{I}\dot{\varphi}_1 - \frac{C_e}{I}(\dot{\varphi}_1 - \dot{\varphi}_2), M^*_{D2} = -\frac{C_2}{I}\dot{\varphi}_2 - \frac{C_e}{I}(\dot{\varphi}_2 - \dot{\varphi}_1)$, where $C_1\dot{\varphi}_1$ and $C_2\dot{\varphi}_2$ are the moments of resistance to viscous air, $C_e(\dot{\varphi}_1 - \dot{\varphi}_2), C_e(\dot{\varphi}_2 - \dot{\varphi}_1)$ are the damping moments created by the elastic element; I is the moment of inertia, $I = 4ms^2$; s is the distance between the center of mass of the pendulum and the axis of rotation ($l = 2s$); m is the mass of the larger pendulum. Since, in the system under consideration, characteristics of various natures can be small (mass ratio, dissipation, etc.), for later use of the multiple scales method, we introduce a formal small parameter ε that only emphasizes the smallness of certain terms in the equations of motion. In calculations, this parameter is taken equal to 1. In addition, C_1, C_2, C_e are constant coefficients that do not depend on the dynamics of the system. In the model, we use a Pade–approximation of the magnetic influence in this form:

$$M_{mag} = \left(a_0 + \frac{a_1\varphi + a_2\varphi^3}{1 + b_1\varphi^2 + b_2\varphi^4}\right) sign\ \varphi, \quad (2)$$

where a_0, a_1, a_2, b_1, b_2 are the coefficients of the model obtained using the procedure of the nonlinear least squares method (NLSM) in order to best satisfy the experimental data presented in Fig. 2. To construct a Fig.2, the parameter of the intensity of the magnetic effect $\gamma = \frac{1}{5}$.



Figire 2. Experimental data of the magnetic moment compared with the numerically matched M_{mag} model.

The coefficients presented in Table 1 were used to construct the approximate model.

Table 1. The value of the coefficients of the magnetic interaction model.

	a_0	a_1	a_2	b_1	b_2
$\varphi > 0$	0.085	2.222484	−49.60483	−19.537113	853.7762
$\varphi < 0$	−0.085	2.222462	−49.604415	−19.5384	853.81384

Later we will use the following abbreviations: $C_{1,2}^{*}=\frac{C_{1,2}}{I}, C_{e}^{*}=\frac{C_{e}}{I}, r^{*}=\frac{mgs}{I}, k_{l}^{*}=\frac{k_{l}}{I}$.

3. Investigation of the localized mode of the pendulum system

3.1. The study of the localized mode by the multiple scales method

The localized mode of the system (1) can be presented analytically after the following time transformation:

$$t \to \sqrt{\varepsilon}t \tag{3}$$

Without reducing the generality, the time designation and differentiation symbols are not changed. Then the following system is considered:

$$\begin{cases} \mu\ddot{\varphi}_1 = \varepsilon\gamma M_{mag_1}^{*} - \varepsilon C_1^{*}\dot{\varphi}_1 - \varepsilon C_e^{*}(\dot{\varphi}_1 - \dot{\varphi}_2) - \varepsilon\mu r^{*}\left(\varphi_1 - \frac{\varepsilon}{6}\varphi_1^3\right) - k_l^{*}(\varphi_1 - \varphi_2), \\ \frac{\ddot{\varphi}_2}{\varepsilon} = \varepsilon\gamma M_{mag_2}^{*} - \varepsilon C_2^{*}\dot{\varphi}_2 - \varepsilon C_e^{*}(\dot{\varphi}_2 - \dot{\varphi}_1) - r^{*}\left(\varphi_2 - \frac{\varepsilon}{6}\varphi_2^3\right) - k_l^{*}(\varphi_2 - \varphi_1). \end{cases} \tag{4}$$

The multiple scales method (MSM) [8-10] is used. We introduce here the following time scales:

$$T_0 = \tau,\ T_1 = \varepsilon\tau, \tau = \omega_0 t, \tag{5}$$

where T_0 is fast time, T_1 is slow time, $\omega_0^2 = k_l^{*}$. That is, the desired functions φ_1,φ_2 are functions of two variables (T_0, T_1). Besides, they are presented in the following series of ε:

$$\varphi_1 = \varphi_{10} + \varepsilon\varphi_{11},\ \varphi_2 = \varphi_{20} + \varepsilon\varphi_{21}. \tag{6}$$

Using standard transformations, one has the following systems corresponding to two approximations by the small parameter:

$$\varepsilon^0: \quad \begin{cases} \mu\omega_0^2\frac{\partial^2\varphi_{10}}{\partial T_0^2} = -k_l^{*}(\varphi_{10} - \varphi_{20}), \\ \omega_0^2\frac{\partial^2\varphi_{20}}{\partial T_0^2} = 0. \end{cases} \tag{7}$$

ε^1:

$$\begin{cases} \mu\omega_0^2\left(\frac{2\partial^2\varphi_{10}}{\partial T_0\partial T_1} + \frac{\partial^2\varphi_{11}}{\partial T_0^2}\right) = \gamma M_{mag_1}^{*} - C_1^{*}\frac{\partial\varphi_{10}}{\partial T_0} - C_e^{*}\left(\frac{\partial\varphi_{10}}{\partial T_0} - \frac{\partial\varphi_{20}}{\partial T_0}\right) - \mu r^{*}\varphi_{10} - k_l^{*}(\varphi_{11} - \varphi_{21}), \\ \omega_0^2\left(\frac{2\partial^2\varphi_{20}}{\partial T_0\partial T_1} + \frac{\partial^2\varphi_{21}}{\partial T_0^2}\right) = -r^{*}\varphi_{20} - k_l^{*}(\varphi_{20} - \varphi_{10}). \end{cases} \tag{8}$$

The solution of the equations (8) is $\varphi_{20} = 0$ and $\varphi_{10} = A_1 cos(T_0 + \vartheta)$, where A_1 and ϑ are functions of the slow scale T_1. The magnetic moment is presented by the shortened Fourier series as

$$M_{mag_1}^{*}(\varphi_{10}) \approx \frac{g_0}{2} + g_1 cos(T_0 + \vartheta) + g_2\cos 2(T_0 + \vartheta) + \cdots + g_6\cos 6(T_0 + \vartheta), \tag{9}$$

where $g_i = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} sign\,(\varphi_{10})\left(a_0 + \frac{a_1\varphi_{10} + a_2\varphi_{10}^3}{1 + b_1\varphi_{10}^2 + b_2\varphi_{10}^4}\right)\cos i(T_0 + \vartheta)\,dT_0, i = \overline{0,6}$.

To prevent the occurrence of the secular terms in equations (8), we exclude the terms containing the multipliers $cos(T_0 + \vartheta)$ and $sin(T_0 + \vartheta)$ in the right sides of these equations, and the following modulation equations are obtained:

$$2\mu\omega_0^2 A_1\frac{\partial\vartheta}{\partial T_1} + \frac{\gamma g_1}{I} - \mu r^{*}A_1 - k_l^{*}\mu A_1 = 0,$$

$$2\mu\omega_0^2\frac{\partial A_1}{\partial T_1} + C_1^{*}A_1 + C_e^{*}A_1 = 0. \tag{10}$$

One has from here that $A_1 = e^{\frac{-(C_1^*+C_e^*)T_1}{2k_l^*}+A_3}$, $\vartheta = \frac{(k_l^*+r^*)T_1}{2\omega_0^2} - \frac{\frac{\gamma g_1}{I} e^{\frac{(C_1^*+C_e^*)T_1}{2k_l^*}}}{(C_1^*+C_e^*)e^{A_3}}$,

where A_3 is an arbitrary constant that sets the initial deviation of the pendulum. Then a solution of the system (8), that is the second approximation by the small parameter is constructed. Comparing the analytical solution with the numerical solution of the basic system (1) obtained by the Runge-Kutta method of the 4th order is made when the initial values of the variables are determined from the analytical solution, shows a good exactness of the analytical approximation when initial angles of the pendulums not exceed such deviations approximately equal to 60°.

3.2. Investigation of the influence of the system parameters and initial conditions on the localized mode

Please, note that here and later a dimension of the system parameters are taken as follows: the mass *m(kg)*, (for the following parameters *m* stands for meter) *s(m)*, $\boldsymbol{k_l}\left(\frac{Nm}{rad}\right)$, $\boldsymbol{C_{1,2,e}}\left(\frac{Nms}{rad}\right)$. Everywhere $\boldsymbol{g} = \mathbf{9.81}\left(\frac{m}{s^2}\right)$. Without loss of generality we shall assume that in all simulations $\dot{\boldsymbol{\varphi}}_{\mathbf{1}}(\mathbf{0}) = \dot{\boldsymbol{\varphi}}_{\mathbf{2}}(\mathbf{0}) = \mathbf{0}$. Besides, in all simulations the formal small parameter ε is equal to 1.

Now let's study the influence of the ratio of the masses of two pendulums μ, $\mu \in [0.01,1]$, while other parameters of the system are chosen as follows: $k_l = 1, m = 0.5, s = 2.5, mgs = 12.2625, I = 12.5, C_1 = 3.1 \cdot 10^{-6}, C_2 = 7.2 \cdot 10^{-6}, C_e = 13.736 \cdot 10^{-6}$. The simulation time here and later is equal to 3000 sec. As a result, one has that an increase in the coefficient μ first increases the deviations near the mode shape (Fig.3a) and then reduces them as it is shown in Fig. 3b (a similar transition is carried out more than once). Then, when the parameter under study is no longer small, the smooth transient to the out-of-phase vibration mode may occur. The Fig. 3 shows the dynamics near the localized mode for $A_3 = 0.7, \varphi_1(0) = -0.0774\, rad$ $(-4.435°), \varphi_2(0) = 0.00334\, rad$ $(0.2°)$ and two different values of the parameter μ.

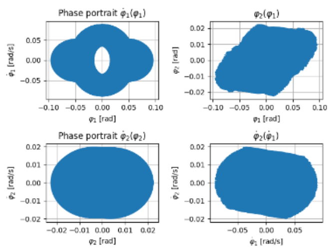


Fig. 3a ($\mu = 0.06$)

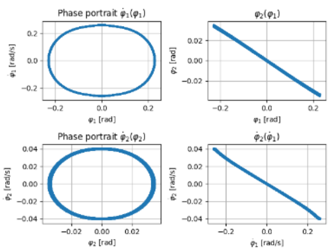


Fig. 3b ($\mu = 0.15$)

Figure 3. Phase portraits, trajectories in the configuration space of two pendulums at different μ at sufficiently small initial angles.

An increase in the coupling coefficient k_l very often (but not always) leads to a stabilization of the localized or out-of-phase mode. We can see in Fig. 4a essential wanderings near such mode shape for small values of the initial angles of the pendulums but some small change of the parameter k_l together with increase the initial angles can reduce these wanderings (Fig. 4b). Other parameters are fixed, namely,, $\mu = 0.31, m = 0.5, s = 2.5, mgs = 12.2625, I = 12.5, C_1 = 3.1 \cdot 10^{-6}, C_2 = 7.2 \cdot 10^{-6}, C_e = 13.736 \cdot 10^{-6}$. Here one has $A_3 = -0.5$, $\varphi_1(0) = -0.018\, rad$ $(-1.031°)$, $\varphi_2(0) = 0.004\, rad$ $(0.23°)$, $k_l = 0.65$ in Fig.4a and $A_3 = -0.5$, $\varphi_1(0) = 0.589\, rad$ $(33.75°)$, $\varphi_2(0) = -0.185\, rad$ $(-10.61°)$, $k_l = 0.7$ in Fig. 4b.

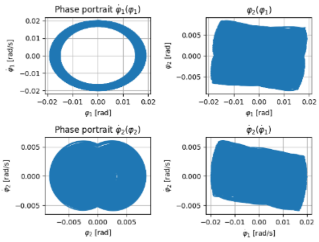


Fig. 4a

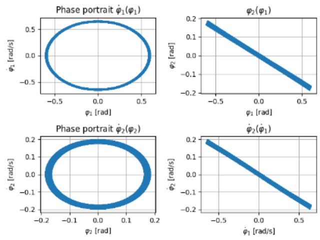


Fig. 4b

Figure 4. Phase portraits and trajectories in the configuration space for different values of the connection between the pendulums.

The distance from the axis of rotation of the pendulum to the center of mass was considered from the range $s \in [0.1, 3]$. The case when the parameter s is equal to 2.5 is shown in Fig. 5. Here $m = 0.5, C_1 = 3.1 \cdot 10^{-6}, C_2 = 7.2 \cdot 10^{-6}, C_e = 13.736 \cdot 10^{-6}$. Besides, $A_3 = -0.5$, $\varphi_1(0) = -0.6015\, rad\ (-34.46°)$, $\varphi_2(0) = 0.1843\, rad\ (10.56°), \mu = 0.31, k_l = 0.6$ in Fig. 5a and $A_3 = 0.7$, $\varphi_1(0) = -0.23\, rad\ (-13.14°)$, $\varphi_2(0) = 0.035\, rad\ (2°)$, $\mu = 0.15$, $k_l = 1$ in Fig. 5b.

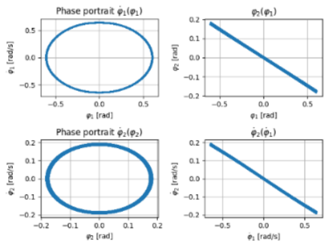


Fig. 5a

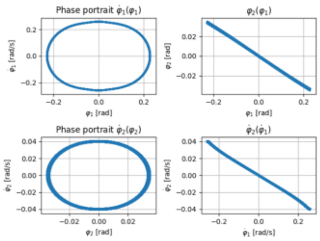


Fig. 5b

Figure 5. Phase portraits and trajectories in the configuration space at $s = 2.5$.

When the coefficient of dissipation changes, localization is not always present. In this part of the study, the values of $C_1, C_2, C_e \in [10^{-6}, 1]$. Here $m = 0.5, A_3 = 0.7$, $\mu = 0.01, k_l = 1, s = 2.5$. Besides, $\varphi_1(0) = -0.8267\, rad\ (-47.42°)$, $\varphi_2(0) = 0.00827\, rad\ (0.4741°)$, $C_1 = 1.35 \cdot 10^{-5}$, $C_2 = 8.5 \cdot 10^{-6}, C_e = 9 \cdot 10^{-5}$ in Fig. 6a and $\varphi_1(0) = 0.2818\, rad\ (16.146°)$, $\varphi_2(0) = -0.003\, rad\ (-0.1719°)$, $C_1 = 3.85 \cdot 10^{-5}$, $C_2 = 3.35 \cdot 10^{-5}, C_e = 0.00034$ in Fig. 6b.
By fixing the system parameters and changing only the parameter $A_3 \in [-15,1]$ which is depended from the initial angles of the pendulums, then we can see that the localized mode is not present in the entire range of initial conditions. The mode is not implemented at small initial angles of inclination of the pendulums.

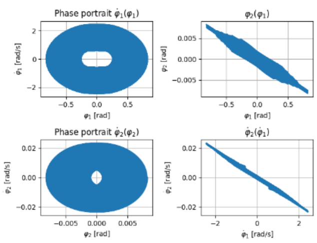


Fig. 6a

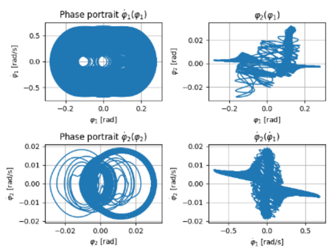


Fig. 6b

Fig. 6. Phase portraits and trajectories in the configuration space of two pendulums with different coefficients of dissipation.

4. Investigation of the coupled mode of the pendulum system

4.1. Investigation of the coupled mode by the multiple scales method

We introduce here also the time scales ($T_0 = \tau$, $T_1 = \varepsilon\tau, \tau = \omega_0 t$, where $\omega_0^2 = r^*$), and use standard transformations of the MSM. Substituting the series (6) to the basic system (1) transformed in accordance with relations (3) and grouping the summands by degrees of ε, one obtains the following:

$$\varepsilon^0: \quad \begin{cases} -k_l^*(\varphi_{10}-\varphi_{20}) = 0, \\ \omega_0^2 \dfrac{\partial^2 \varphi_{20}}{\partial T_0^2} = -r^*\varphi_{20} - k_l^*(\varphi_{20}-\varphi_{10}). \end{cases} \qquad 11)$$

$$\varepsilon^1: \quad \begin{cases} \mu\omega_0^2 \dfrac{\partial^2\varphi_{10}}{\partial T_0^2} = \gamma M^*_{mag_1} - C_1^* \dfrac{\partial\varphi_{10}}{\partial T_0} - C_e^*\left(\dfrac{\partial\varphi_{10}}{\partial T_0} - \dfrac{\partial\varphi_{20}}{\partial T_0}\right) - \mu r^*\varphi_{10} - k_l^*(\varphi_{11}-\varphi_{21}), \\ \omega_0^2\left(\dfrac{2\partial^2\varphi_{20}}{\partial T_0 \partial T_1} + \dfrac{\partial^2\varphi_{21}}{\partial T_0^2}\right) = \gamma M^*_{mag_2} - C_2^*\dfrac{\partial\varphi_{20}}{\partial T_0} - C_e^*\left(\dfrac{\partial\varphi_{20}}{\partial T_0} - \dfrac{\partial\varphi_{10}}{\partial T_0}\right) - r^*\left(\varphi_{21} - \dfrac{\varphi_{20}^3}{6}\right) - k_l^*(\varphi_{21}-\varphi_{11}). \end{cases} \qquad (12)$$

As we can see that the coupled vibration mode is the in-phase one, and in zero approximation one has the following equality: $\varphi_{10} = \varphi_{20} = A_1 cos(T_0 + \vartheta)$.

The magnetic moment for two pendulums $M_{mag_{1,2}}$ is decomposed into a Fourier series as before (for the second magnetic action we use the coefficients $h_i, i = \overline{(1,6)}$). Like to the sub-Section 3.1 the modulation equations are here the following:

$$2\omega_0^2 A_1 \frac{\partial\vartheta}{\partial T_1} + \frac{\gamma h_1}{I} + \frac{r^* A_1^3}{8} + \mu\omega_0^2 A_1 + \frac{\gamma g_1}{I} - \mu r^* A_1 = 0,$$

$$2\omega_0^2 \frac{\partial A_1}{\partial T_1} + C_1^* A_1 + C_2^* A_1 = 0. \qquad (13)$$

One has that

$$A_1 = e^{\frac{-(C_1^*+C_2^*)T_1}{2r^*} + A_3}, \vartheta = \frac{-\frac{\gamma}{I}(h_1+g_1)e^{\frac{(C_1^*+C_2^*)T_1}{2r^*}}}{(C_1^*+C_2^*)e^{A_3}} + \frac{r^* e^{2A_3}}{16(C_1^*+C_2^*)e^{\frac{(C_1^*+C_2^*)T_1}{r^*}}}, \qquad (14)$$

where A_3an arbitrary constant that sets the initial deviation of the pendulum. Then a solution of the second approximation by the small parameter is obtained.

Comparing the analytical solution with the numerical solution of the basic system (1) obtained by the Runge-Kutta method show a good exactness of the analytical approximation for small values of the parameter μ, and not large values of the initial angles of the pendulums like to such conclusions presented in sub-Section 3.1.

4.2. Investigation of the influence of system parameters and initial conditions on the dynamics of the system (coupled mode case)

Let's start the study with the influence of the initial conditions: we will vary the parameter $A_3 \in [-1.5, 1.5]$. This corresponds to a change in the initial angle of the pendulum from -60 to 60 degrees. Let's fix other system parameters as follows: $\mu = 0.05, \gamma = \frac{1}{5}, k_l = 1, m = 0.5, s = 2.5, mgs = 12.2625, I = 12.5, C_1 = 3.1 \cdot 10^{-5}, C_2 = 7.2 \cdot 10^{-5}, C_e = 13.736 \cdot 10^{-5}$. The simulation time is equal here to 3000 s. A stable coupled mode is observed at the following intervals of the initial deviations of the angle φ_1 (Fig. 7). In the diagram, we can see how the initial angle changes depending on the parameter value. Analogous diagram can be obtained for the angle φ_2.

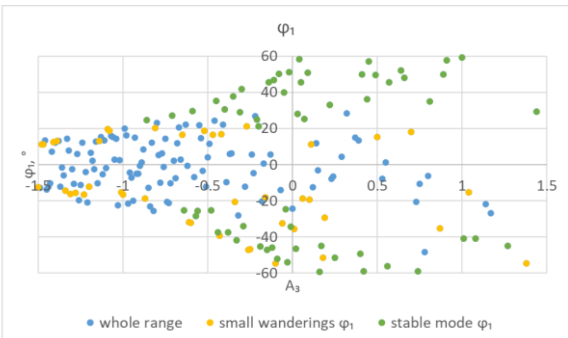


Figure 7. Comparison of the entire range of initial conditions (blue dots) with the values of the initial angle $\varphi_1(0)(°)$ of the pendulums at which there is a stable coupled mode (green dots).

We can observe that an increase in the value of the pendulum masses ratio leads to a clearer representation of the vibration mode trajectories in both configuration and phase spaces as it is shown in Fig. 8. Here $A_3 = -0.84,\ \varphi_1(0) = -0.4062\, rad\ (-23.27°),\ \varphi_2(0) = -0.4035\, rad\ (-23.12°),$ $\mu = 0.05$ in Fig. 8a and $A_3 = -0.84,\ \varphi_1(0) = -0.4062\, rad\ (-23.27°),\ \varphi_2(0) = -0.4035\, rad\ (-23.12°),\ \mu = 0.15$ in Fig. 8b.

Increasing the coupling coefficient of the pendulums leads to the emergence of a stable coupled regime, when the masses of the pendulums differ and the distance from the axis of rotation to the center of mass of the pendulum is significantly. For the Fig. 9, the system parameters are fixed as follows: $m = 0.5,\ C_1 = 3.1 \cdot 10^{-5},\ C_2 = 7.2 \cdot 10^{-5},\ C_e = 13.736 \cdot 10^{-5}$. Here $A_3 = 0.75, \varphi_1(0) = 0.6142\, rad\ (35.191°), \varphi_2(0) = 0.6333\, rad\ (36.285°), \mu = 0.55, s = 1.5,\ k_l = 0.1$ in Fig. 9a, and $A_3 = 0.75, \varphi_1(0) = 0.6265\, rad\ (35.9°), \varphi_2(0) = 0.6333\, rad\ (36.285°), \mu = 0.55, s = 1.5, k_l = 0.28$ in Fig. 9b.

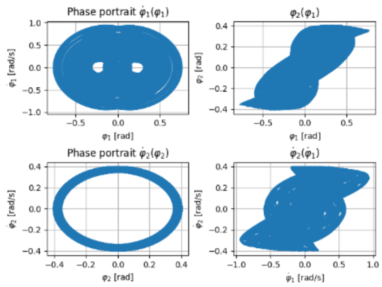


Fig. 8a

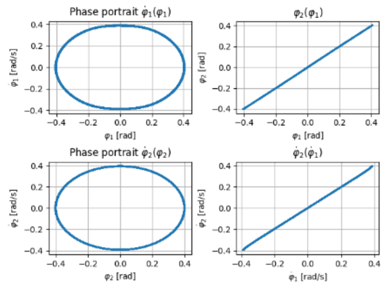


Fig. 8b

Figure 8. Phase portraits and trajectories in configuration space under various initial conditions and mass proportionality coefficients.

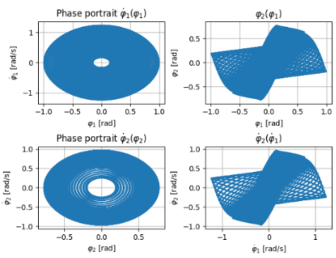


Fig. 9a

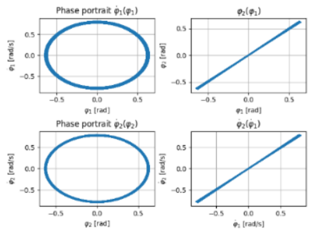


Fig. 9b

Figure 9. Phase portraits and trajectories in the configuration space when changing the ratio of the pendulum masses and the coupling coefficient.

An increase in the distance from the axis of rotation to the center of mass of the pendulums in the interval $s \in [0.1,4]$ leads to a stable coupled mode when the parameter μ is not too small, and the connection between the pendulums is significant. The greater the coefficient of proportionality of the masses of two pendulums, as well as the distance from the axis of rotation to the center of mass and the connection between the pendulums, the less wandering of the trajectories near the in-phase form. But even under these conditions, the mode is not always stable: it can be seen that with increasing distance, initially the wanderings decrease, and then the wanderings increase and the mode disappears. After that, the system gradually returns to the stable coupled mode. In Fig. 10 $m = 0.5, C_1 = 3.1 \cdot 10^{-5}, C_2 = 7.2 \cdot 10^{-5}, C_e = 13.736 \cdot 10^{-5}$. In addition, $A_3 = -0.69, \varphi_1(0) = -0.1333\, rad\ (-7.64°), \varphi_2(0) = -0.139\, rad\ (-7.964°), \mu = 0.02,\ k_l = 1, s = 2.5$ in Fig.10a, and $A_3 = -0.69,\ \varphi_1(0) = 0.4974\, rad\ (28.5°),\ \varphi_2(0) = 0.49863\, rad\ (28.57°), \mu = 0.25, k_l = 1, s = 3$ in Fig. 10b.

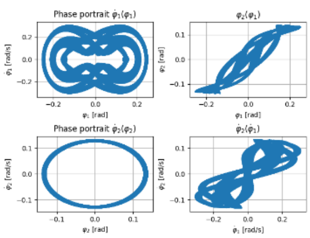


Fig. 10a

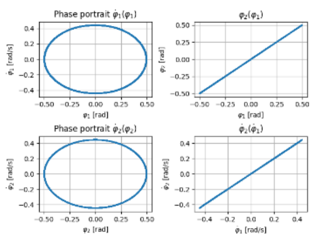


Fig. 10b

Figure 10. Changing the connected mode trajectories when changing the distance from the axis of rotation to the center of mass.

5. Conclusions

Analysis of the localized and connected modes in the system of the coupled pendulums with different inertial characteristics in a magnetic field shows that these modes do not exist in the entire range of the pendulum initial angles. In particular, an irregular behaviour is observed for small initial angles of the pendulums when the magnetic moment is dominant. An increase in the mass ratio of the pendulums may lead to a decrease in the wandering near of localized mode trajectory. An increase in the coupling coefficient between the pendulums usually does not lead to the manifestation of a stable localized mode. Besides, an increase of this coefficient leads to the appearance of a stable coupled (in-phase) mode, provided that the ratio of masses of the pendulums is not very small. In this case, when the connection between the pendulums is not small, an increase in the distance from the axis of rotation to the center of mass of the pendulums leads to the appearance of the stable coupled mode.

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