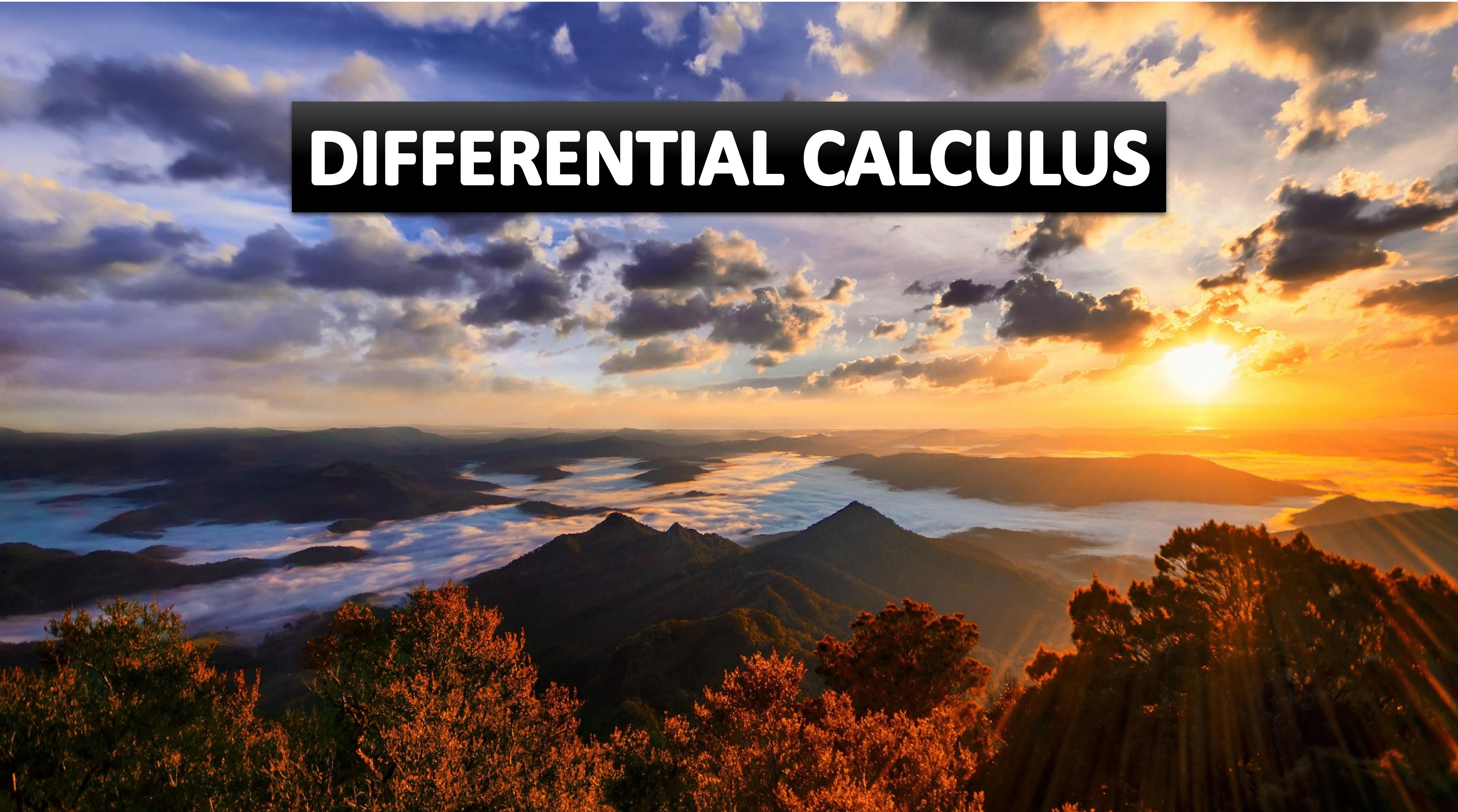
DIFFERENTIAL CALCULUS



DIFFERENTIAL CALCULUS is a branch of mathematics that deals with **RATES OF CHANGE.**

The **INSTANTANEOUS RATE OF CHANGE** of a function…at a particular point…can be found by calculating the gradient of the tangent…to the function at that point.

More generally…for a function $f(x)$ we can define a **DERIVATIVE FUNCTION…OR…GRADIENT FUNCTION** $\boldsymbol{f'(x)}$ which allows us to calculate the gradient of the tangent at any point on the function.

The process of finding the derivative function is called **DIFFERENTIATION.**

To differentiate a function…
We can use the theory of LIMITS.

Limits

If $f(x)$ can be made as close as we like to some real number…A…by making x sufficiently close to…(but not equal to) a…then we say that $f(x)$ has a **LIMIT**…of A…as x approaches a…and we write…

$$\lim_{x \to a} f(x) = A$$

$$the\ limit\ as\ x\ approaches\ a\ of\ f(x) equals\ to\ A$$

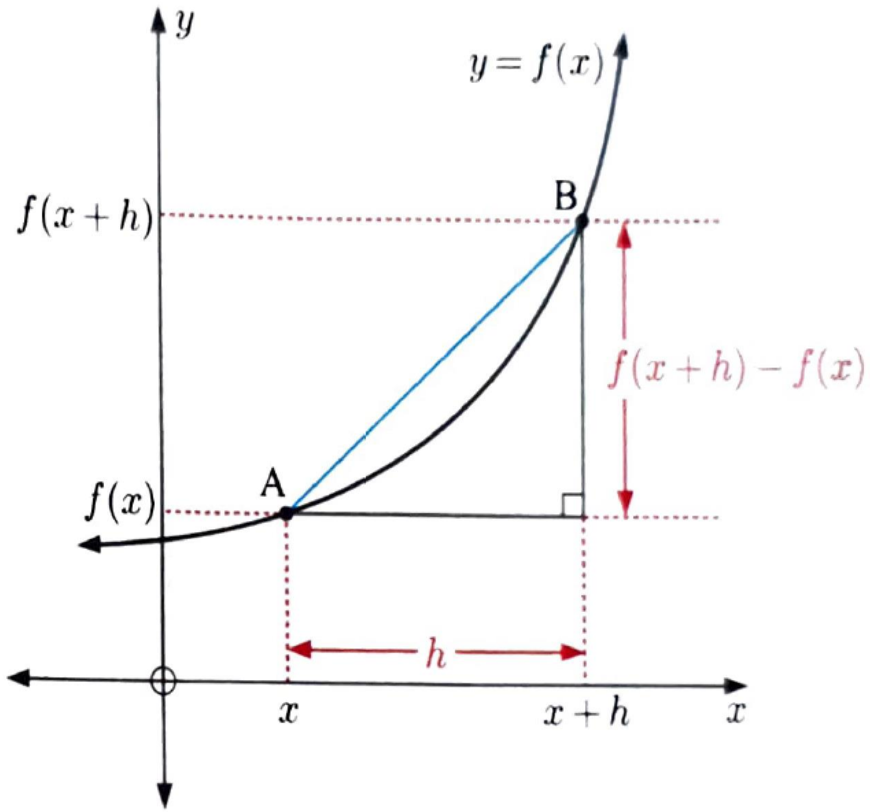
In this case…$f(x)$ is said to **CONVERGE TO A**…as x approaches a

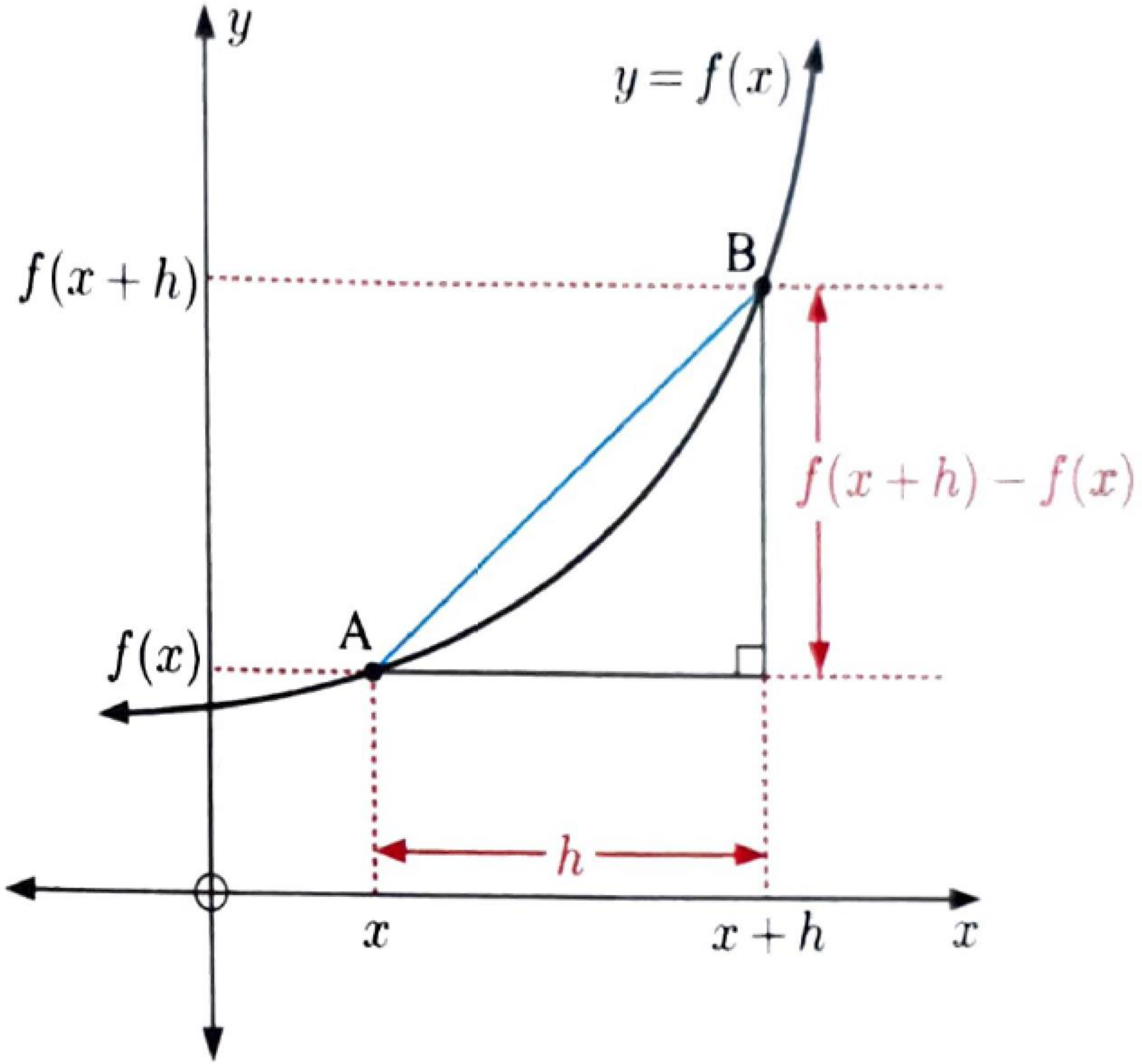
Find the derivative function by 1st principles

Consider the function $y = f(x)$ where A is the point $(x, f(x))$....and B is the point $(x+h, f(x+h))$

The gradient of the chord [AB] is:

$$\frac{f(x+h) - f(x)}{h}$$





y
$y = f(x)$
$f(x+h)$
B
$f(x+h) - f(x)$
A
$f(x)$
h
x
$x+h$
x

Another way to express this gradient is using the LEIBNIZ NOTATION:

$$\frac{\delta y}{\delta x}$$

Where $\delta y = f(x + h) - f(x)$…is the vertical step

And $\delta x = h$ is the horizontal step

$$\delta y = delta\ y = change\ in\ y$$

To calculate the gradient of the tangent at A…we let the point B get closer and closer to A.

This means that the horizontal step $\delta x = h$ becomes infinitely small…and the gradient of the tangent at the general point A $(x, f(x))$ is: $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

The derivative function of $y = f(x)$ is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{dy}{dx}$$

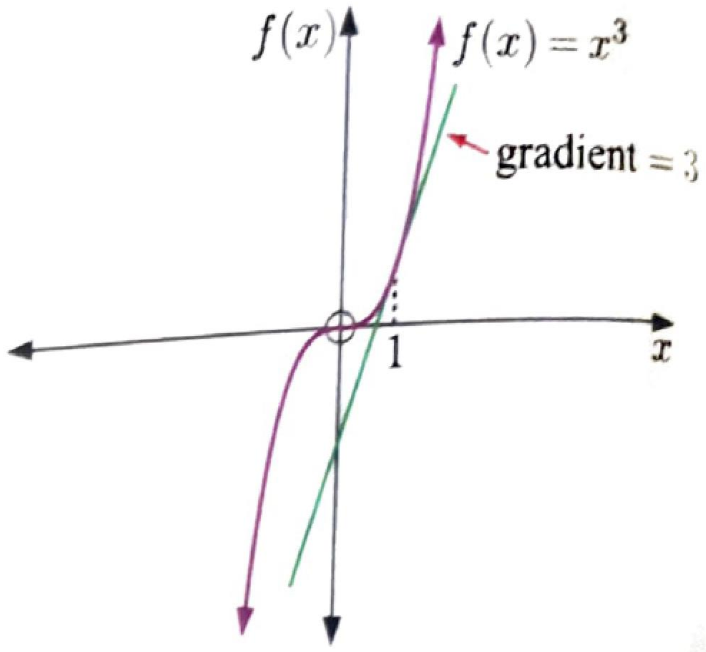
$$read\ as \ldots dee\ y \ldots by\ dee\ x \ldots$$

Example 1

Use 1st principles to find the derivative of $f(x) = x^3$

Find $f'(1)$ and interpret your answer

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\
&= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h} \\
&= \lim_{h\to 0}(3x^2+3xh+h^2) \\
&= 3x^2
\end{aligned}
$$



$$f'(1) = 3(1)^2 = 3$$

Therefore the gradient of the tangent to $f(x) = x^3$ at the point where $x = 1$ is 3

$$f(x) = x^2$$

Find the derivative from 1st principles!!!

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} \frac{h(2x + h)}{h} \\
&= 2x + h \ldots sub\ h = 0 \\
&= 2x + (0) \\
&= 2x
\end{aligned}$$

$$f(x) = 2x \ldots (2)$$

Find the derivative from 1st principles!!!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h) - 2x}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x} + 2h - \cancel{2x}}{h}$$

$$= \lim_{h \to 0} \frac{2h}{h}$$

$$= \frac{\cancel{h}(2)}{\cancel{h}}$$

$$= 2$$

$$f(x) = 2x^2 \dots (4x)$$

Find the derivative from 1st principles!!!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{2\left(x^2 + 2xh + h^2\right) - 2x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{- 2x^2}}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h)}{\cancel{h}}$$

$$= 4x + 2h \dots sub\ h = 0$$

$$4x$$

Example 2

Given $f(x) = \frac{1}{x}$…use 1st principles to find $f'(x)$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \times \left(\frac{x(x+h)}{x(x+h)} \right) \\
&= \lim_{h \to 0} \frac{x - (x+h)}{hx(x+h)} \\
&= \lim_{h \to 0} \frac{-h}{hx(x+h)} \\
&= -1/x^2
\end{aligned}
$$

EXERCISE 2A

1 a Use first principles to find the derivative function $f'(x)$ for $f(x) = x^2$.

b Find $f'(3)$ and interpret your answer.

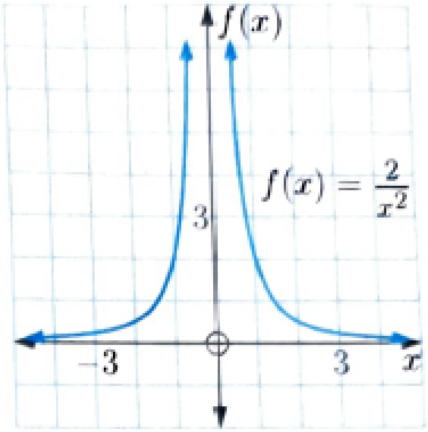
2 Find $f'(x)$ from first principles, given that $f(x)$ is:

a x b 1 c $-x^2$ d $2x^3$

3 Find $f'(x)$ from first principles, given that $f(x)$ is:

a $\frac{3}{x}$ b $\frac{1}{x^2}$ c $-\frac{2}{x^3}$

4 The graph of $f(x) = \frac{2}{x^2}$ is shown alongside.



a Use first principles to find $f'(x)$.

b Hence, find the gradient of the tangent to $f(x)$ at:

i $x = -1$ ii $x = 2$.

c Copy the graph, and include the information from b.

a)

$$f(x) = x$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h}$$

$$= \lim_{h\to 0} \frac{x + h - x}{h}$$

$$= \lim_{h\to 0} \frac{h}{h}$$

$$= 1$$

b)

$$f(x) = 1$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The Constant rule says
the derivative of
any constant function is always 0.

c)

$$f(x) = -x^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h)^2 - (-x^2)}{h}$$

$$= \lim_{h \to 0} \frac{-\left(x^2 + 2xh + h^2\right) + x^2}{h}$$

$$= \lim_{h \to 0} \frac{-x^2 - 2xh - h^2 + x^2}{h}$$

$$= \lim_{h \to 0} \frac{-2xh - h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(-2x - h)}{h}$$

$$= -2x - h \ldots .sub\ h = 0$$

$$= -2x$$

d)

$$f(x) = 2x^3$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^3 - 2x^3}{h}$$

$$= \lim_{h \to 0} \frac{2\left(x^3 + 3x^2h + 3xh^2 + h^3\right) - 2x^3}{h}$$

$$= \lim_{h \to 0} \frac{2x^3 + 6x^2h + 6xh^2 + h^3 - 2x^3}{h}$$

$$= \lim_{h \to 0} \frac{6x^2h + 6xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{h(6x^2 + 6xh + h^3)}{h}$$

$$= 6x^2 + 6xh + h^3 \ldots sub\ h = 0$$

$$= 6x^2$$

a)

$$f(x)=\frac{3}{x}$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{3}{(x+h)}-(\frac{3}{x})}{h}$$

$$=\lim_{h\to 0}\left(\frac{3x}{x(x+h)}-\frac{3(x+h)}{x(x+h)}\right).\frac{1}{h}$$

$$=\lim_{h\to 0}\left(\frac{3x-3x-3h}{x^2+xh}\right).\frac{1}{h}$$

$$=\lim_{h\to 0}\left(\frac{-3h}{x^2+xh}\right).\frac{1}{h}$$

$$=\lim_{h\to 0}\frac{-3}{x^2+xh}\ldots sub\ h=0$$

$$=-3/x^2$$

b)

$$f(x) = 1/x^2$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right) . \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}\right) . \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{x^2 - \left(x^2 + 2xh + h^2\right)}{x^2(x+h)^2}\right) . \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{x^2 - x^2 - 2xh - h^2}{x^2(x+h)^2}\right) . \frac{1}{h}$$

$$= \lim_{h\to 0} \left(\frac{-2xh - h^2}{x^2(x+h)^2}\right) . \frac{1}{h}$$

$$= \left(\frac{-2x - h}{x^2(x+h)^2}\right)$$

$$= \left(\frac{-2x - h}{x^4 - 2x^3h + x^2h^2}\right)$$

$$= \frac{-2x}{x^4}$$

$$= -\frac{2}{x^3}$$

$$f(x) = -\frac{2}{x^3}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \left(-\frac{2}{(x+h)^3} - \left(-\frac{2}{x^3} \right) \right) . \frac{1}{h}$$

$$= \lim_{h \to 0} \left(-\frac{2}{x^3 + 3x^2h + 3xh^2 + h^3} + \frac{2}{x^3} \right) . \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{-2(x^3)}{x^3(x^3 + 3x^2h + 3xh^2 + h^3)} + \frac{2(x^3 + 3x^2h + 3xh^2 + h^3}{x^3(x^3 + 3x^2h + 3xh^2 + h^3)} \right) . \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{-2x^3 + 2x^3 + 6x^2h + 6xh^2 + 2h^3}{x^6 + 3x^5h + 3x^4h^2 + x^3h^3} \right) . \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{6x^2h + 6xh^2 + 2h^3}{x^6 + 3x^5h + 3x^4h^2 + x^3h^3} \right) . \frac{1}{h}$$

$$= \left(\frac{6x^2}{x^6} \right)$$

$$= 6x^{-4}$$

Example 3

Given $f(x) = \sqrt{x}$ ...use 1st principles to find $f'(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$f'(x) = \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right) \times \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$f'(x) = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$Let\ y = x^3 + 4x \ldots find\ \frac{dy}{dx}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 + 4(x+h) - (x^3 + 4x)}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 4x + 4h - x^3 - 4x}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 + 4}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 + 3xh + h^2 + 4}{h}$$

$$= 3x^2 + 4$$

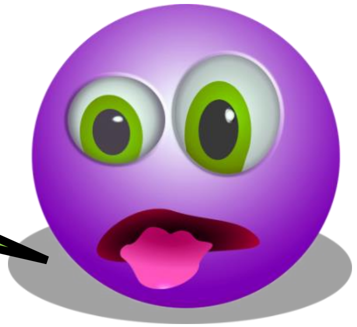
So here’s a function…

$$y = 3x^3$$

The derivative is…

$$y = 9x^2$$

How did I get it…?



This is how…

$\times$

$$y = 3x^3$$

-1

$3 \times 3 = 9$

$3 - 1 = 2$

$$\boxed{y = 9x^2}$$

Remember…the derivative gives you the rate of change.

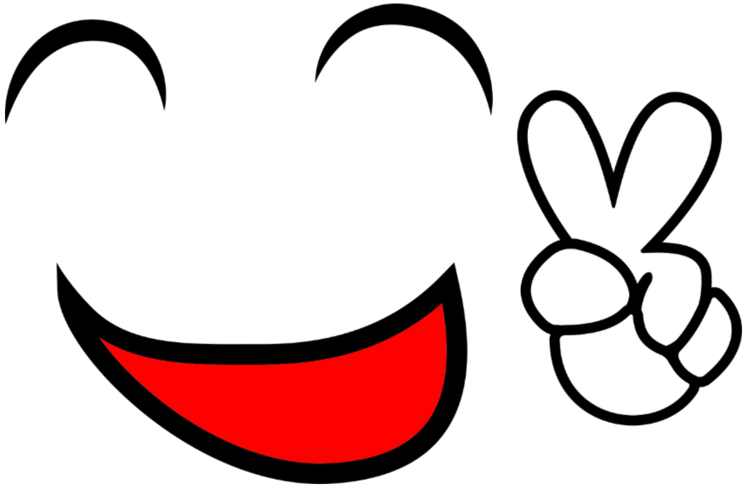
The rate of change is calculated based on the graph that is drawn.

The derivative is the gradient of the tangent to the curve at some point on the curve.

Remember that the tangent touches the curve ONLY AT 1 POINT!

Simple rules of differentiation

$f(x)$	$f'(x)$	Name of rule
C (constant)	0	Differentiating a constant
x^n	nx^{n-1}	Differentiating x^n
$cu(x)$	$cu'(x)$	Constant times a function
$u(x) + v(x)$	$u'(x) + v'(x)$	Addition rule



Example 4

Find $f'(x)$ for $f(x)$ equal to:

a) $2x^3 + 7x^2 - 5x + 4$

b) $\frac{5}{x} - \frac{6}{x^2}$

---

a)

$$f(x) = 2x^3 + 7x^2 - 5x + 4$$
$$f'(x) = 2(3x^2) + 7(2x) - 5(1)$$
$$= 6x^2 + 14x - 5$$

b)

$$f(x) = \frac{5}{x} - \frac{6}{x^2}$$
$$= 5x^{-1} - 6x^{-2}$$
$$f'(x) = 5(-1x^{-2}) - 6(-2x^{-3})$$
$$= -5x^{-2} + 12x^{-3}$$
$$= -\frac{5}{x^2} + \frac{12}{x^3}$$

EXERCISE 2B

1 Find $f'(x)$ given that $f(x)$ is:

a x^5

b x^9

c $3x^6$

d $-2x^7$

e $\frac{1}{2}x^4$

f $x^3 + 4x$

g $x^2 - 6x + 2$

h $4x^2 + 3x - 5$

i $8x^2 - 1$

j $9 - 3x^2$

k $x^3 - 2x + 5$

l $\frac{1}{3}x^3 - 5x^2 + 4x$

m $6 - 5x - 2x^2$

n $x^6 + \frac{1}{3}x^4 - 7x$

o $x^2(x - 4)$

p $(x + 3)(x - 1)$

2 Find $\dfrac{dy}{dx}$ for:

a $y = \dfrac{1}{x^3}$

b $y = \dfrac{1}{x^4}$

c $y = \dfrac{1}{x^7}$

d $y = \dfrac{4}{x}$

e $y = -\dfrac{2}{x^5}$

f $y = \dfrac{5}{x^2}$

g $y = 3x + \dfrac{2}{x}$

h $y = \dfrac{1}{x} - \dfrac{4}{x^2}$

i $y = \dfrac{3}{x^2} - \dfrac{7}{x^4}$

j $y = \dfrac{1}{2x}$

k $y = -\dfrac{2}{3x^2}$

l $y = 7x^2 - \dfrac{1}{4x^5}$

m $y = \dfrac{x^2 + 6}{x}$

n $y = \dfrac{x - 5}{x^3}$

o $y = \dfrac{x^2 - 4x + 3}{x^2}$

1.

a)

$$y = x^5$$
$$y' = 5x^4$$

b)

$$y = x^9$$
$$y' = 9x^8$$

c)

$$y = 3x^6$$
$$y' = 18x^5$$

d)

$$y = -2x^7$$
$$y' = -14x^6$$

e)

$$y = \frac{1}{2}x^4$$
$$y' = 2x^3$$

f)

$$y = x^3 + 4x$$
$$y' = 3x^2 + 4$$

g)

$$y = x^2 - 6x + 2$$
$$y' = 2x - 6$$

h)

$$y = 4x^2 + 3x - 5$$
$$y' = 8x + 3$$

i)

$$y = 8x^2 - 1$$
$$y' = 16x$$

j)

$$y = 9 - 3x^2$$
$$y' = -6x$$

k)

$$y = x^3 - 2x + 5$$
$$y' = 3x^2 - 2$$

l)

$$y = \frac{1}{3}x^3 - 5x^2 + 4x$$
$$y' = x^2 - 10x + 4$$

m)

$$y = 6 - 5x - 2x^2$$
$$y' = -5 - 4x$$

n)

$$y = x^6 + \frac{1}{3}x^4 - 7x$$
$$y' = 6x^5 + \frac{4}{3}x^3 - 7$$

o)

$$y = x^2(x - 4)$$
$$y = x^3 - 4x^2$$
$$y' = 3x^2 - 8x$$

p)

$$y = (x + 3)(x - 1)$$
$$= x^2 - x + 3x - 3$$
$$= x^2 + 2x - 3$$
$$= 2x + 2$$

2.

a)

$$y = \frac{1}{x^3}$$
$$= x^{-3}$$
$$y' = -3x^{-4}$$

b)

$$y = \frac{1}{x^4}$$
$$= x^{-4}$$
$$y' = -4x^{-5}$$

c)

$$y = \frac{1}{x^7}$$
$$= x^{-7}$$
$$y' = -7x^{-8}$$

d)

$$y = \frac{4}{x}$$
$$= 4x^{-1}$$
$$y' = -4x^{-2}$$

e)

$$y = -\frac{2}{x^5}$$
$$= -2x^{-5}$$
$$y' = 10x^{-6}$$

f)

$$y = \frac{5}{x^2}$$
$$= 5x^{-2}$$
$$y' = -10x^{-3}$$

g)

$$y = 3x + \frac{2}{x}$$
$$= 3x + 2x^{-1}$$
$$y' = 3 - 2x^{-2}$$

h)

$$y = \frac{1}{x} - \frac{4}{x^2}$$
$$= x^{-1} - 4x^{-2}$$
$$y' = -x^{-2} + 8x^{-3}$$

i)

$$y = \frac{3}{x^2} - \frac{7}{x^4}$$
$$= 3x^{-2} - 7x^{-4}$$
$$y' = -6x^{-3} + 28x^{-5}$$

j)

$$y = \frac{1}{2x}$$
$$= \frac{1}{2}x^{-1}$$
$$y' = -\frac{1}{2}x^{-2}$$

k)

$$y = -\frac{2}{3x^2}$$
$$= -\frac{2}{3}x^{-2}$$
$$y' = \frac{4}{3}x^{-3}$$

l)

$$y = 7x^2 - \frac{1}{4x^5}$$
$$= 7x^2 - \frac{1}{4}x^{-5}$$
$$y' = 14x + \frac{5}{4}x^{-6}$$

m)

$$y = \frac{x^2+6}{x}$$
$$= x + \frac{6}{x}$$
$$= x + 6x^{-1}$$
$$y' = 1 - 6x^{-2}$$

n)

$$y = \frac{x-5}{x^3}$$
$$= x^{-2} - \frac{5}{x^3}$$
$$= x^{-2} - 5x^{-3}$$
$$y' = -2x^{-3} + 15x^{-4}$$

o)

$$y = \frac{x^2-4x+3}{x^2}$$
$$= -\frac{4x}{x^2} + \frac{3}{x^2}$$
$$= -\frac{4}{x} + \frac{3}{x^2}$$
$$= -4x^{-1} + 3x^{-2}$$
$$y' = 4x^{-2} - 6x^{-3}$$

Example 5

Find $f'(x)$ for $f(x)$ equal to:

$a)\quad 6\sqrt{x}+\frac{3}{x}$

$b)\quad \frac{9}{\sqrt[3]{x}}$

---

a)

$$f(x)=6\sqrt{x}+\frac{3}{x}$$
$$=6x^{\frac{1}{2}}+3x^{-1}$$
$$f'(x)=6\left(\frac{1}{2}x^{-\frac{1}{2}}\right)+3(-1x^{-2})$$
$$=3x^{-\frac{1}{2}}-3x^{-2}$$
$$=\frac{3}{\sqrt{x}}-\frac{3}{x^2}$$

b)

$$f(x)=\frac{9}{\sqrt[3]{x}}$$
$$=9x^{-\frac{1}{3}}$$
$$f'(x)=9(-\frac{1}{3}x^{-\frac{4}{3}})$$
$$=-3x^{-\frac{4}{3}}$$
$$=-\frac{3}{\sqrt[3]{x^4}}$$

$$f(x) = 6x^3 - 9x + 4$$

$$f(x) = 2t^4 - 10t^2 + 13t$$

$$f(x) = 4x^7 - 3x^{-7} + 9x$$

$$f(x) = y^{-4} - 9y^{-3} + 8y^{-2} + 12$$

$$f(x) = 10\sqrt[3]{x} - 2\sqrt[4]{x}$$

$$f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3$$

$$f(t) = \frac{4}{t} - \frac{1}{6t^3} + \frac{8}{t^5}$$

Find the derivative of these examples…may the force be with you…

A few more examples…

$$f(x) = \frac{6}{\sqrt{x^3}} + \frac{1}{8x^4} - \frac{1}{3x^{10}}$$

$$y = x\left(3x^2 - 9\right)$$

$$f(x) = (x-4)(2x+x^2)$$

$$h(x) = \frac{4x^3 - 7x + 8}{x}$$

$$f(y) = \frac{y^5 - 5y^3 + 2y}{y^3}$$

$$f(x) = 6x^3 - 9x + 4$$
$$f'(x) = 18x^2 - 9$$

$$f(x) = 2t^4 - 10t^2 + 13t$$
$$f'(x) = 8t^3 - 20t + 13$$

$$f(x) = 4x^7 - 3x^{-7} + 9x$$
$$f'(x) = 28x^6 + 21x^{-8} + 9$$

$$f(x) = y^{-4} - 9y^{-3} + 8y^{-2} + 12$$
$$f'(x) = -4y^{-5} + 27y^{-4} - 16y^{-3}$$

$$f(x) = 10\sqrt[3]{x} - 2\sqrt[4]{x}$$
$$f'(x) = \frac{10}{3}x^{-\frac{2}{3}} + \frac{1}{2}x^{-\frac{3}{4}}$$

$$f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3$$
$$f'(x) = 6x^{\frac{-2}{5}} - \frac{7}{2}x^{\frac{5}{2}} + 16x^{\frac{5}{3}}$$

$$f(t) = \frac{4}{t} - \frac{1}{6t^3} + \frac{8}{t^5}$$
$$= 4t^{-1} - \frac{1}{6}t^{-3} + 8t^{-5}$$
$$f'(t) = -4t^{-2} + \frac{1}{2}t^{-4} - 40t^{-6}$$

$$f(x) = \frac{6}{\sqrt{x^3}} + \frac{1}{8x^4} - \frac{1}{3x^{10}}$$

$$= 6x^{-\frac{3}{2}} + \frac{1}{8}x^{-4} - \frac{1}{3}x^{-10}$$

$$f'(x) = -9x^{-\frac{5}{2}} - \frac{1}{2}x^{-5} + \frac{10}{3}x^{-11}$$

$$y = x\left(3x^2 - 9\right)$$

$$= 3x^3 - 9x$$

$$\frac{dy}{dx} = 9x^2 - 9$$

$$f(x) = (x-4)(2x+x^2)$$
$$= 2x^2 + x^3 - 8x - 4x^2$$
$$f'(x) = -4x + 3x^2 - 8$$

$$h(x) = \frac{4x^3 - 7x + 8}{x}$$
$$= 4x^2 - 7 + \frac{8}{x}$$
$$h'(x) = 8x - 8x^{-2}$$

$$f(y) = \frac{y^5 - 5y^3 + 2y}{y^3}$$
$$= y^2 - 5 + 2y^{-2}$$
$$f'(y) = 2y - 4y^{-3}$$

Find derivatives for the following examples…

a) $y = x^4$

b) $y = 3x^3$

c) $y = 5x^2 - 8$

d) $y = x^2 - 2x + 7$

e) $y = x^3 - 2x^2$

f) $y = x^6 - 5x^4 - 4x^3 - 2x$

g) $y = 8x^{1.25}$

h) $y = x^{\frac{1}{2}} - 3x^{-2}$

i) $y = \frac{3x^2+6x^3}{x}$

j) $y = \frac{1}{4}x^4(x^2 - 12)$

k) $y = -3x^{-\frac{5}{3}}$

l) $y = 2x^{-1} - 6x^{-2} + 4x^{-3}$

m) $y = \sqrt[6]{x}$

n) $y = x + x.\sqrt{x}$

o) $y = \frac{2x^5+6x^2-3}{2x^2}$

p) $y = \frac{x^2-3x}{\sqrt{x}}$

a)

$$y = x^4$$
$$y' = 4x^3$$

b)

$$y = 3x^3$$
$$y' = 9x^2$$

c)

$$y = 5x^2 - 8$$
$$y' = 10x$$

d)

$$y = x^2 - 2x + 7$$
$$y' = 2x - 2$$

e)

$$y = x^3 - 2x^2$$
$$y' = 3x^2 - 4x$$

f)

$$y = x^6 - 5x^4 - 4x^3 - 2x$$
$$y' = 6x^5 - 20x^3 - 12x^2 - 2$$

g)

$$y = 8x^{1.25}$$
$$y' = 10x^{\frac{1}{4}}$$

h)

$$y = x^{\frac{1}{2}} - 3x^{-2}$$
$$y' = \frac{1}{2}x^{-\frac{1}{2}} + 6x^{-3}$$

i)

$$y = \frac{3x^2 + 6x^3}{x}$$
$$y = 3x + 6x^2$$
$$y' = 3 + 12x$$

j)

$$y = \frac{1}{4}x^4(x^2 - 12)$$
$$= \frac{1}{4}x^6 - 3x^4$$
$$y' = \frac{3}{2}x^5 - 12x^3$$

k)

$$y = -3x^{-\frac{5}{3}}$$
$$y' = 5x^{-\frac{8}{3}}$$

l)

$$y = 2x^{-1} - 6x^{-2} + 4x^{-3}$$
$$y' = -2x^{-2} + 12x^{-3} - 12x^{-4}$$

m)

$$y = \sqrt[6]{x}$$
$$= x^{\frac{1}{6}}$$
$$y' = \frac{1}{6}x^{-\frac{5}{6}}$$

n)

$$y = x + x.\sqrt{x}$$
$$= x + x.x^{\frac{1}{2}}$$
$$= x + x^{\frac{3}{2}}$$
$$y' = \frac{3}{2}x^{\frac{1}{2}}$$

o)

$$y = \frac{2x^5 + 6x^2 - 3}{2x^2}$$
$$= \frac{2x^5}{2x^2} + \frac{6x^2}{2x^2} - \frac{3}{2x^2}$$
$$= x^3 + 3 - \frac{3}{2}x^{-2}$$
$$y' = 3x^2 + 3x^{-3}$$

p)

$$y = \frac{x^2 - 3x}{\sqrt{x}}$$
$$= \frac{x^2}{x^{\frac{1}{2}}} - \frac{3x}{x^{\frac{1}{2}}}$$
$$= x^{3/2} - 3x^{\frac{1}{2}}$$
$$= \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-\frac{1}{2}}$$

3 Find $f'(x)$ for $f(x)$ equal to:

a $\sqrt{x}$

b $4\sqrt{x}$

c $\sqrt[3]{x}$

d $x\sqrt{x}$

e $\dfrac{2}{\sqrt{x}}$

f $2x + 8\sqrt{x}$

g $6x^2\sqrt{x}$

h $\dfrac{7}{x} - \dfrac{4}{\sqrt{x}}$

i $\dfrac{3x-8}{\sqrt{x}}$

j $\dfrac{6}{\sqrt[3]{x}}$

k $-\dfrac{3}{x^2\sqrt{x}}$

l $x^2 - \dfrac{5}{x\sqrt{x}}$

3. a)

$$y = \sqrt{x}$$
$$= x^{\frac{1}{2}}$$
$$y' = \frac{1}{2}x^{-\frac{1}{2}}$$

b)

$$y = 4.\sqrt{x}$$
$$= 4x^{\frac{1}{2}}$$
$$y' = 2x^{-\frac{1}{2}}$$

c)

$$\sqrt[3]{x}$$
$$= x^{\frac{1}{3}}$$
$$y' = \frac{1}{3}x^{-\frac{2}{3}}$$

d)

$$y = x.\sqrt{x}$$
$$= x^{\frac{3}{2}}$$
$$y' = \frac{3}{2}x^{\frac{1}{2}}$$

e)

$$y = \frac{2}{\sqrt{x}}$$
$$= 2x^{-\frac{1}{2}}$$
$$y' = -1x^{-\frac{3}{2}}$$

f)

$$y = 2x + 8\sqrt{x}$$
$$= 2x + 8x^{\frac{1}{2}}$$
$$y' = 2 + 4x^{-\frac{1}{2}}$$

g)

$$y = 6x^2.\sqrt{x}$$
$$= 6x^2.x^{\frac{1}{2}}$$
$$= 6x^{\frac{5}{2}}$$
$$y' = 15x^{\frac{3}{2}}$$

h)

$$y = \frac{7}{x} - \frac{4}{\sqrt{x}}$$
$$= 7x^{-1} - 4x^{-\frac{1}{2}}$$
$$y' = -7x^{-2} + 2x^{-\frac{3}{2}}$$

i)

$$y = \frac{3x - 8}{\sqrt{x}}$$
$$= 3x^{\frac{1}{2}} - 8x^{-\frac{1}{2}}$$
$$y' = \frac{3}{2}x^{-\frac{1}{2}} + 4x^{-\frac{3}{2}}$$

j)

$$y = \frac{6}{\sqrt[3]{x}}$$
$$= 6x^{-\frac{1}{3}}$$
$$y' = -2x^{-\frac{4}{3}}$$

The chain rule

We can often write complicated functions as the composite of two or more simpler functions.

For example

$$y = \left(x^2 + 3x\right)^4$$

This could be written as…

$$y = u^4 \ldots where\ u = x^2 + 3x$$

$$or\ as\ \ldots$$

$$y = g\big(f(x)\big)\ \mathrm{where}\ \mathrm{g(x)} = \mathrm{x}^4\ \mathrm{f(x)} - \mathrm{x}^2 + 3\mathrm{x}$$

Example 6

a) $g\big(f(x)\big)$ if $g(x) = \sqrt{x}$ and $f(x) = 2 - 3x$

b) $g(x)$ and $f(x)$ such that $g\big(f(x)\big) = \frac{1}{x-x^2}$

---

a)

$$g(f(x))$$
$$= g(2 - 3x)$$
$$= \sqrt{2 - 3x}$$

b)

$$if\ we\ let\ f(x) = x - x^2 \ldots then \ldots$$

$$g\big(f(x)\big) = \frac{1}{f(x)}$$

$$g(x) = \frac{1}{x}\ and\ f(x) = x - x^2$$

EXERCISE 2C

1 Find $g(f(x))$ if:

a $g(x) = x^2$ and $f(x) = 2x + 7$

b $g(x) = 2x + 7$ and $f(x) = x^2$

c $g(x) = \sqrt{x}$ and $f(x) = 3 - 4x$

d $g(x) = 3 - 4x$ and $f(x) = \sqrt{x}$

e $g(x) = \dfrac{2}{x}$ and $f(x) = x^2 + 3$

f $g(x) = x^2 + 3$ and $f(x) = \dfrac{2}{x}$

2 Find $g(x)$ and $f(x)$ such that $g(f(x))$ is:

a $(3x + 10)^3$

b $\dfrac{1}{2x + 4}$

c $\sqrt{x^2 - 3x}$

d $\dfrac{10}{(3x - x^2)^3}$

1. a)

$$g(x) = x^2 \ldots f(x) = 2x + 7$$
$$g(f(x)) = (2x + 7)^2$$

b)

$$g(x) = 2x + 7 \ldots f(x) = x^2$$
$$g(f(x)) = 2x^2 + 7$$

c)

$$g(x) = \sqrt{x} \ldots f(x) = 3 - 4x$$
$$g(f(x)) = \sqrt{3 - 4x}$$

d)

$$g(x) = 3 - 4x \ldots f(x) = \sqrt{x}$$
$$g(f(x)) = 3 - 4\sqrt{x}$$

e)

$$g(x) = \frac{2}{x} \ldots f(x) = x^2 + 3$$
$$g(f(x)) = \frac{2}{x^2 + 3}$$

f)

$$g(x) = x^2 + 3 \ldots f(x) = \frac{2}{x}$$
$$g(f(x)) = \left(\frac{2}{x}\right)^2 + 3$$
$$= \frac{4}{x^2} + 3$$

2. a)

$$(3x+10)^3$$
$$g(x)=x^3$$
$$f(x)=3x+10$$

b)

$$\frac{1}{2x+4}$$
$$g(x)=\frac{1}{x}$$
$$f(x)=2x+4$$

d)

$$\sqrt{x^2-3x}$$
$$g(x)=\sqrt{x}$$
$$f(x)=x^2-3x$$

e)

$$\frac{10}{(3x-x^2)^2}$$
$$g(x)=\frac{10}{x^3}$$
$$f(x)=3x-x^2$$

More examples of the chain rule

a)

$$y = \left(6x^2 + 7x\right)^4$$
$$u^4 = 4u^3$$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= 4\left(6x^2 + 7x\right)^3 . (12x + 7)$$

b)

$$y = \left(4t^2 - 3t + 2\right)^{-2}$$
$$u^{-2} = -2u^{-3}$$
$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= -2\left(4t^2 - 3t + 2\right)^{-3} . (8t - 3)$$

c)

$$y = \sqrt[3]{1 - 8x}$$
$$= (1 - 8x)^{\frac{1}{3}}$$
$$u^{\frac{1}{3}} = \frac{1}{3}u^{-\frac{2}{3}}$$
$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= \frac{1}{3}(1 - 8x)^{-\frac{2}{3}} . (-8)$$

d)

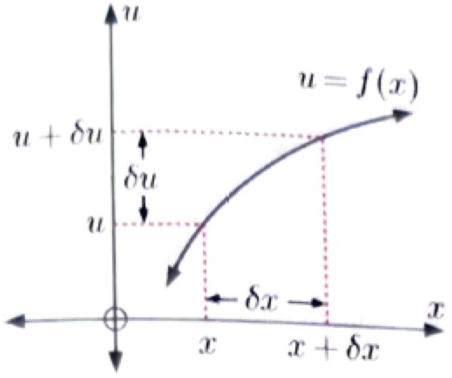
$$y = \csc(7x)$$
$$\csc(u) = -\csc(u)\cot(u)$$
$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$-\csc(7x) . \cot(7x) . (7)$$
$$-7\csc(7x) . \cot(7x)$$

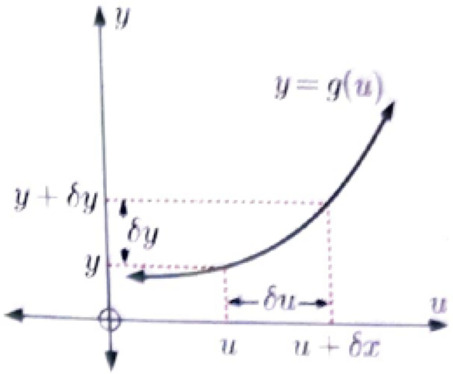
Derivatives of composite functions

The reason we are interested in writing complicated functions as composite functions is to make finding derivatives easier.

Consider $y = g(u)$ where $u = f(x)$.

For a small change of δx in x, there is a small change of $f(x + \delta x) - f(x) = \delta u$ in u and a small change of δy in y.





Now $\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta u} \times \dfrac{\delta u}{\delta x}$ {fraction multiplication}

As $\delta x \to 0$, $\delta u \to 0$ also.

$$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \quad \text{\{limit rule\}}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 7

Find $\frac{dy}{dx}$ if:

$a)\ \ y = \left(x^2 - 2x\right)^4$

$b)\ \ y = \frac{4}{\sqrt{1-2x}}$

a)

$$y = \left(x^2 - 2x\right)^4$$
$$y = u^4 \ldots where\ u = x^2 - 2x$$
$$now \ldots \frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= 4u^3(2x - 2)$$
$$= 4\left(x^2 - 2x\right)^3(2x - 2)$$

$$y = \frac{4}{\sqrt{1 - 2x}}$$
$$y = 4u^{-\frac{1}{2}} \ldots where\ u = 1 - 2x$$
$$now \ldots \frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= 4 \times (-\frac{1}{2}u^{-\frac{3}{2}}) \times (-2)$$
$$= 4u^{-\frac{3}{2}}$$
$$= 4(1 - 2x)^{-\frac{3}{2}}$$

EXERCISE 2C.2

1 Write in the form au^n, clearly stating what u is:

a $\dfrac{1}{(2x-1)^2}$

b $\sqrt{x^2-3x}$

c $\dfrac{2}{\sqrt{2-x^2}}$

d $\sqrt[3]{x^3-x^2}$

e $\dfrac{4}{(3-x)^3}$

f $\dfrac{10}{x^2-3}$

2 Differentiate $y=(2x+3)^2$ by:

a using the chain rule with $u=2x+3$

b expanding $y=(2x+3)^2$ then differentiating term-by-term.

1. a)

$$y = \frac{1}{(2x-1)^2}$$

$$let\ u = 2x - 1$$

$$y = \frac{1}{u^2}$$

$$= u^{-2}$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$y' = -2u^{-3}.(2)$$

$$= -2(2x-1)^{-3}.(2)$$

$$= -4(2x-1)$$

b)

$$y = \sqrt{x^2 - 3x}$$

$$= \left(x^2 - 3x\right)^{\frac{1}{2}}$$

$$let\ u = x^2 - 3x$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$y = u^{\frac{1}{2}}$$

$$y' = \frac{1}{2}u^{-\frac{1}{2}}.(2x-3)$$

$$= \frac{1}{2}\left(x^2 - 3x\right)^{-\frac{1}{2}}.(2x-3)$$

c)

$$y = \frac{2}{\sqrt{2 - x^2}}$$

$$= \frac{2}{(2 - x^2)^{\frac{1}{2}}}$$

$$let\ u = 2 - x^2$$

$$y = \frac{2}{u^{\frac{1}{2}}}$$

$$= 2u^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$y' = -u^{-\frac{3}{2}}.(-2x)$$

$$= -\left(2 - x^2\right)^{-\frac{3}{2}}.(-2x)$$

$$2x\left(2 - x^2\right)^{-\frac{3}{2}}$$

d)

$$y = \sqrt[3]{x^3 - x^2}$$
$$= \left(x^3 - x^2\right)^{\frac{1}{3}}$$
$$let\ u = x^3 - x^2$$

$$y = u^{\frac{1}{3}}$$
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$y' = \frac{1}{3}u^{-\frac{2}{3}}.(3x^2 - 2x)$$
$$= \frac{1}{3}\left(x^3 - x^2\right)^{-\frac{2}{3}}.(3x^2 - 2x)$$

e)

$$y = \frac{4}{(3 - x)^3}$$
$$let\ u = 3 - x$$
$$y = \frac{4}{u^3}$$
$$= 4u^{-3}$$

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= -12u^{-4}.(-1)$$
$$= -12(3 - x)^{-4}.(-1)$$
$$= 12(3 - x)^{-4}$$

f)

$$y = \frac{10}{x^2 - 3}$$
$$let\ u = x^2 - 3$$

$$y = 10u^{-1}$$
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$y' = -10u^{-2}.(2x)$$
$$= -10\left(x^2 - 3\right)^{-2}.(2x)$$

$$y = (2x+3)^2$$
$$let\ u = 2x+3$$
$$y = u^2$$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$y' = 2u.(2)$$
$$= 2(2x+3).(2)$$
$$= 4(2x+3)$$
$$= 8x+12$$

$$y = (2x+3)^2$$
$$= (2x+3)(2x+3)$$
$$= 4x^2 + 6x + 6x + 9$$
$$= 4x^2 + 12x + 9$$

$$y' = 8x+12$$

3 Find the derivative function $\dfrac{dy}{dx}$ for:

a $y = (4x - 5)^2$

b $y = \dfrac{1}{5 - 2x}$

c $y = \sqrt{3x - x^2}$

d $y = (1 - 3x)^4$

e $y = 6(5 - x)^3$

f $y = \sqrt[3]{2x^3 - x^2}$

g $y = \dfrac{6}{(5x - 4)^2}$

h $y = (x^2 - 5x + 8)^5$

i $y = 2\left(x^2 - \dfrac{2}{x}\right)^3$

3.

a)

$$y = (4x-5)^2$$
$$= u^2$$
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= 2(4x-5).(4)$$
$$= 8(4x-5)$$

b)

$$y = \frac{1}{5-2x}$$
$$= \frac{1}{u}$$
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= -5(5-2x)^{-2}.(-2)$$
$$= 2(5-2x)^{-2}$$

c)

$$y = \sqrt{3x-x^2}$$
$$= \left(3x-x^2\right)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= \frac{1}{2}\left(3x-x^2\right)^{-\frac{1}{2}}.(3-2x)$$

d)

$$y = (1-3x)^4$$
$$= u^4$$
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= 4(1-3x)^3.(-3)$$
$$= -12(1-3x)^3$$

e)

$$y = 6(5-x)^3$$
$$= 6x^3$$
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= 18(5-x)^2.(-1)$$
$$= -18(5-x)^2$$

f)

$$y = \sqrt[3]{2x^3-x^2}$$
$$= \left(2x^3-x^2\right)^{\frac{1}{3}}$$
$$= u^{\frac{1}{3}}$$
$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$
$$= \frac{1}{3}\left(2x^3-x^2\right)^{-\frac{2}{3}}.(6x^2-2x)$$

g)

$$y = \frac{6}{(5x-4)^2}$$

$$= \frac{6}{u^2} = 6u^{-2}$$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

$$= -12(5x-4)^{-3} . (5)$$

$$= -60(5x-4)^{-3}$$

h)

$$y = \left(x^2 - 5x + 8\right)^5$$

$$= u^5$$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

$$5\left(x^2 - 5x + 8\right)^4 . (2x-5)$$

i)

$$y = 2\left(x^2 - \frac{2}{x}\right)^3$$

$$= 2\left(x^2 - 2x^{-1}\right)^3$$

$$= 2u^3$$

$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$

$$= 6\left(x^2 - 2x^{-1}\right)^2 . (2x + 2x^{-2})$$

Find the following derivatives…

1. $y = \left(6x^2 + 7x\right)^4$
2. $y = \left(4t^2 - 3t + 2\right)^{-2}$
3. $y = \sqrt[3]{1 - 8x}$
4. $y = \csc(7x)$
5. $y = 2\sin(3x + \tan(x))$
6. $y = \tan(4 + 10x)$
7. $y = 5 + e^{4t+t^7}$
8. $y = e^{1-\cos(x)}$
9. $y = 2^{1-6x}$
10. $y = \ln(1 - 5y^2 + y^3)$

More examples…

1.

$$y = \left(6x^2 + 7x\right)^4$$
$$u^4 = 4u^3$$
$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= 4\left(6x^2 + 7x\right)^3 . (12x + 7)$$

2.

$$y = \left(4t^2 - 3t + 2\right)^{-2}$$
$$u^{-2} = -2u^{-3}$$
$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= -2\left(4t^2 - 3t + 2\right)^{-3} . (8t - 3)$$

3.

$$y = \sqrt[3]{1 - 8x}$$
$$= (1 - 8x)^{-\frac{2}{3}} . (-8)$$

4.

$$y = \csc(7x)$$
$$\csc(u) = -\csc(u) . \cot(u)$$
$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= -\csc(7x) . \cot(7x) . (7)$$
$$= -7\csc(7x) . \cot(7x)$$

5.

$$y = 2\sin(3x + \tan(x))$$
$$2\sin(u)$$
$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= 2\cos(3x + \tan(x)) . 3 + \sec^2(x)$$

6.

$$y = \tan(4 + 10x)$$
$$\tan(u)$$
$$\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$$
$$= \sec^2(4 + 10x) . (10)$$
$$= 10\sec^2(4 + 10x)$$

7.

$$y = 5 + e^{4t+t^7}$$
$$e^u \ldots where\ u = 4t + t^7$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$\left(e^{4t+t^7}\right) . (4 + 7t^7)$$

8.

$$y = e^{1-\cos(x)}$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= e^{1-\cos(x)} . \sin(x)$$

9.

$$y = 2^{1-6x}$$
$$2^u = \ln(2) . 2^{1-6x}$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \ln(2) . 2^{1-6x} . (-6)$$
$$= -6\left(2^{1-6x}\right) . \ln(2)$$

10.

$$y = \ln(1 - 5y^2 + y^3)$$
$$\ln(x) = \frac{1}{x}$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{1}{1 - 5y^2 + y^3} . (-10y + 3y^2)$$
$$= \frac{(-10y + 3y^2)}{(1 - 5y^2 + y^3)}$$

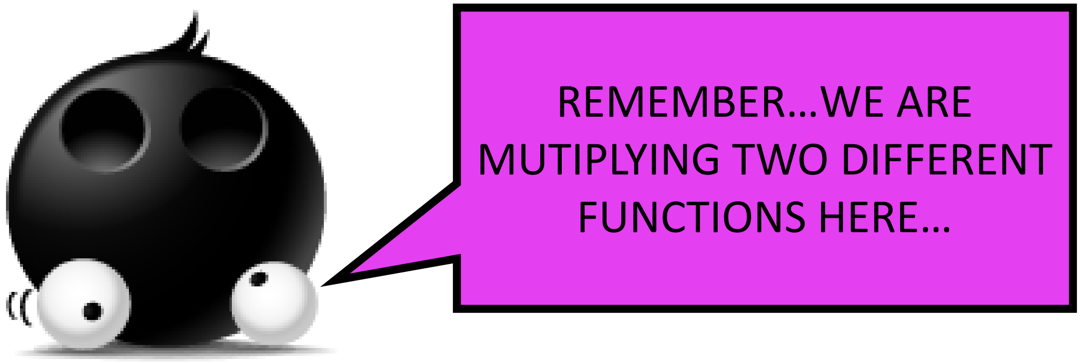
The product rule

If $f(x) = u(x)v(x)$…then…$f'(x) = u'(x).v(x) + u(x).v'(x)$

Alternatively…

If $y = uv \ldots where\ u\ and\ v\ are\ functions\ of\ x \ldots then \ldots$

$$\frac{dy}{dx} = \frac{du}{dx}.v + u.\frac{\mathrm{dv}}{\mathrm{dx}}$$



Example 8

Find $\frac{dy}{dx}$ if…

$a)\quad y = \sqrt{x}(2x+1)^3$

$b)\quad y = x^2\left(x^2 - 2x\right)^4$

---

a)

$$y = \sqrt{x}(2x+1)^3$$

$$is\ the\ product\ of\ u = x^{\frac{1}{2}}\ and\ v = (2x+1)^3$$

$$u' = \frac{1}{2}x^{-\frac{1}{2}} \ldots and\ v' = 3(2x+1)^2 \times 2 \ldots (chain\ rule)$$

$$= 6(2x+1)^2$$

$$now\,\frac{dy}{dx} = u'v + uv'$$

$$= \frac{1}{2}x^{-\frac{1}{2}}.(2x+1)^3 + x^{\frac{1}{2}}.6(2x+1)^2$$

$$\frac{1}{2}x^{-\frac{1}{2}}.(2x+1)^3 + 6x^{\frac{1}{2}}(2x+1)^2$$

$$y = x^2\left(x^2 - 2x\right)^4$$

$$is\ the\ product\ of\ u = x^2 and\ v = \left(x^2 - 2x\right)^4$$

$$u' = 2x \ldots and\ v' = 4\left(x^2 - 2x\right)^3(2x - 2) \ldots (chain\ rule)$$

$$now \ldots \frac{dy}{dx} = u'v + uv'$$

$$= 2x\left(x^2 - 2x\right)^4 + x^2.4\left(x^2 - 2x\right)^3(2x - 2)$$

$$= 2x\left(x^2 - 2x\right)^4 + 4x^2.\left(x^2 - 2x\right)^3(2x - 2)$$

EXERCISE 2D

1 Use the product rule to differentiate:

a $f(x) = x(x-1)$

b $f(x) = 2x(x+1)$

c $f(x) = x^2\sqrt{x+1}$

d $f(x) = (x+3)(x-1)$

e $f(x) = x\sqrt{x^2-1}$

f $f(x) = x(x+1)^2$

2 Find $\frac{dy}{dx}$ using the product rule:

a $y = x^2(2x-1)$

b $y = 4x(2x+1)^3$

c $y = x^2\sqrt{3-x}$

d $y = \sqrt{x}(x-3)^2$

e $y = 5x^2(3x^2-1)^2$

f $y = \sqrt{x}(x-x^2)^3$

1. a)

$$f(x) = x.(x-1)$$
$$f'(x) = (x-1) + x$$
$$= 2x - 1$$

b)

$$f(x) = 2x(x+1)$$
$$f'(x) = 2(x+1) + 2x$$
$$= 2x + 2 + 2x$$
$$= 4x + 2$$

c)

$$f(x) = x^2.\sqrt{x+1}$$
$$= x^2.(x+1)^{\frac{1}{2}}$$
$$f'(x) = 2x.\sqrt{x+1} + \frac{1}{2}(x+1)^{-\frac{1}{2}}.x^2$$

d)

$$f(x) = (x+3)(x-1)$$
$$= x^2 + 3x - x - 3$$
$$= x^2 + 2x - 3$$
$$= 2x + 2$$

e)

$$f(x) = x.\left(x^2 - 1\right)^{\frac{1}{2}}$$
$$f'(x) = \left(x^2 - 1\right)^{\frac{1}{2}} + \frac{1}{2}\left(x^2 - 1\right)^{-\frac{1}{2}}.2x + x$$

f)

$$f(X) = x(x+1)^2$$
$$f'(x) = (x+1)^2 + 2(x+1)(x)$$
$$= (x+1)^2 + 2x(x+1)$$

2.

a)

$$y = x^2(2x-1)$$
$$\frac{dy}{dx} = 2x(2x-1) + (2)\left(x^2\right)$$
$$= 2x(2x-1) + 2x^2$$

b)

$$y = 4x(2x+1)^3 \ldots chain\ rule \ldots$$
$$\frac{dy}{dx} = 4(2x+1)^3.(2x+1)^3 + 6(2x+1)^2.(4x)$$
$$= 4(2x+1)^3.(2x+1)^3 + 24x(2x+1)^2$$

c)

$$y = x^2.\sqrt{3-x}$$
$$= x^2.(3-x)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 2x(3-x)^{\frac{1}{2}} - \frac{1}{2}(3-x)^{-\frac{1}{2}}.\left(x^2\right)$$
$$= 2x(3-x)^{\frac{1}{2}} - \frac{1}{2}x^2.(3-x)^{-\frac{1}{2}}$$

d)

$$y = \sqrt{x}(x-3)^2$$
$$= x^{\frac{1}{2}}.(x-3)^2$$
$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}.(x-3)^2 + 2(x-3)\left(x^{\frac{1}{2}}\right)$$
$$= \frac{1}{2}x^{-\frac{1}{2}}.(x-3)^2 + 2x^{\frac{1}{2}}.(x-3)$$

e)

$$y = 5x^2\left(3x^2-1\right)^2$$
$$\frac{dy}{dx} = 10x\left(3x^2-1\right)^2 + 6(3x-1)\left(5x^2\right)$$
$$= 10x\left(3x^2-1\right)^2 + 60x^3(3x-1)$$

The Quotient rule

If $Q(x) = \frac{u(x)}{v(x)} \ldots then \ldots$

$$Q'(x) = \frac{u'(x).v(x) - u(x).v'(x)}{[v(x)]^2}$$

Alternatively…

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

Example 9

Use the quotient rule to find:

a) $y = \frac{1+3x}{x^2+1} \ldots u = 1 + 3x \ldots v = x^2 + 1$

b) $y = \frac{\sqrt{x}}{(1-2x)^2} \ldots u = \sqrt{x} = x^{\frac{1}{2}} \ldots v = (1-2x)^2$

$$y = \frac{1+3x}{x^2+1}$$
$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$
$$= \frac{3\left(x^2+1\right) - (1+3x).2x}{(x^2+1)^2}$$
$$= \frac{3x^2 + 3 - 2x - 6x^2}{(x^2+1)^2}$$
$$= \frac{3 - 2x - 3x^2}{(x^2+1)^2}$$

$$y = \frac{\sqrt{x}}{(1-2x)^2}$$
$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$
$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(\mathbf{1} - \mathbf{2x})^2 - x^{\frac{1}{2}} \times (-4(1-2x))}{(1-2x)^4}$$
$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}$$
$$= \frac{(1-2x)[\frac{1-2x}{2\sqrt{x}} + 4\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)]}{(1-2x)^3}$$

$$= \frac{1 - 2x + 8x}{2\sqrt{x}(1-2x)^3}$$
$$= \frac{6x + 1}{2\sqrt{x}(1-2x)^3}$$

EXERCISE 2E

1 Use the quotient rule to find $\frac{dy}{dx}$ if:

a $y = \frac{1+3x}{2-x}$

b $y = \frac{x^2}{2x+1}$

c $y = \frac{x}{x^2-3}$

d $y = \frac{\sqrt{x}}{1-2x}$

e $y = \frac{x^2-3}{3x-x^2}$

f $y = \frac{x}{\sqrt{1-3x}}$

2 Find the gradient of the tangent to:

a $y = \frac{x}{1-2x}$ at $x = 1$

b $y = \frac{x^3}{x^2+1}$ at $x = -1$

c $y = \frac{\sqrt{x}}{2x+1}$ at $x = 4$

d $y = \frac{x^2}{\sqrt{x^2+5}}$ at $x = -2$.

1. a)

$$y = \frac{1+3x}{2-x}$$

$$\frac{dy}{dx} = \frac{3(2-x) - (-1)(1+3x)}{(2-x)^2}$$

$$= \frac{6 - 3x + 1 + 3x}{(2-x)^2}$$

$$= \frac{7}{(2-x)^2}$$

b)

$$y = \frac{x^2}{2x+1}$$

$$\frac{dy}{dx} = \frac{(2x)(2x+1) - (2)(x^2)}{(2x+1)^2}$$

$$= \frac{4x^2 + 2x - 2x^2}{(2x+1)^2}$$

$$= \frac{2x^2 + 2x}{(2x+1)^2}$$

c)

$$y = \frac{x}{x^2-3}$$

$$\frac{dy}{dx} = \frac{(x^2-3) - (2x)(x)}{(x^2-3)^2}$$

$$= \frac{x^2 - 3 - 2x^2}{(x^2-3)^2}$$

$$= \frac{-x^2 - 3}{(x^2-3)^2}$$

e)

$$y = \frac{x^2-3}{3x-x^2}$$

$$\frac{dy}{dx} = \frac{(2x)(3x-x^2) - (3-2x)(x^2-3)}{(3x-x^2)^2}$$

$$= \frac{6x^2 - 2x^3 - 3x^2 + 9 + 2x^3 - 6x}{(3x-x^2)^2}$$

$$= \frac{3x^2 - 6x + 9}{(3x-x^2)^2}$$