EXPONENTIAL GROWTH & DECAY

The exponential function is
One of the most important functions
In the applications of mathematics…

E.G…it is used in the study of
Bacteria growth in a lab,
Decay of radioactive material,
And calculating interest earned
In a bank account…!

If y is a function of t, such that:

$$y\,(t) = ka^t$$

Where $k>0$ & a are constants.

$a>0$ & a cannot be 1

Then we say that y is growing/decaying exponentially.

Growing if: $a>1$

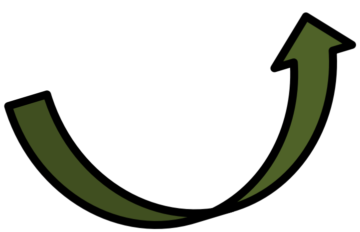
Decaying if: $0<a<1$

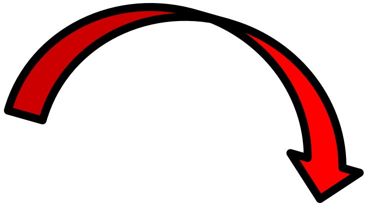
Can also be written as:

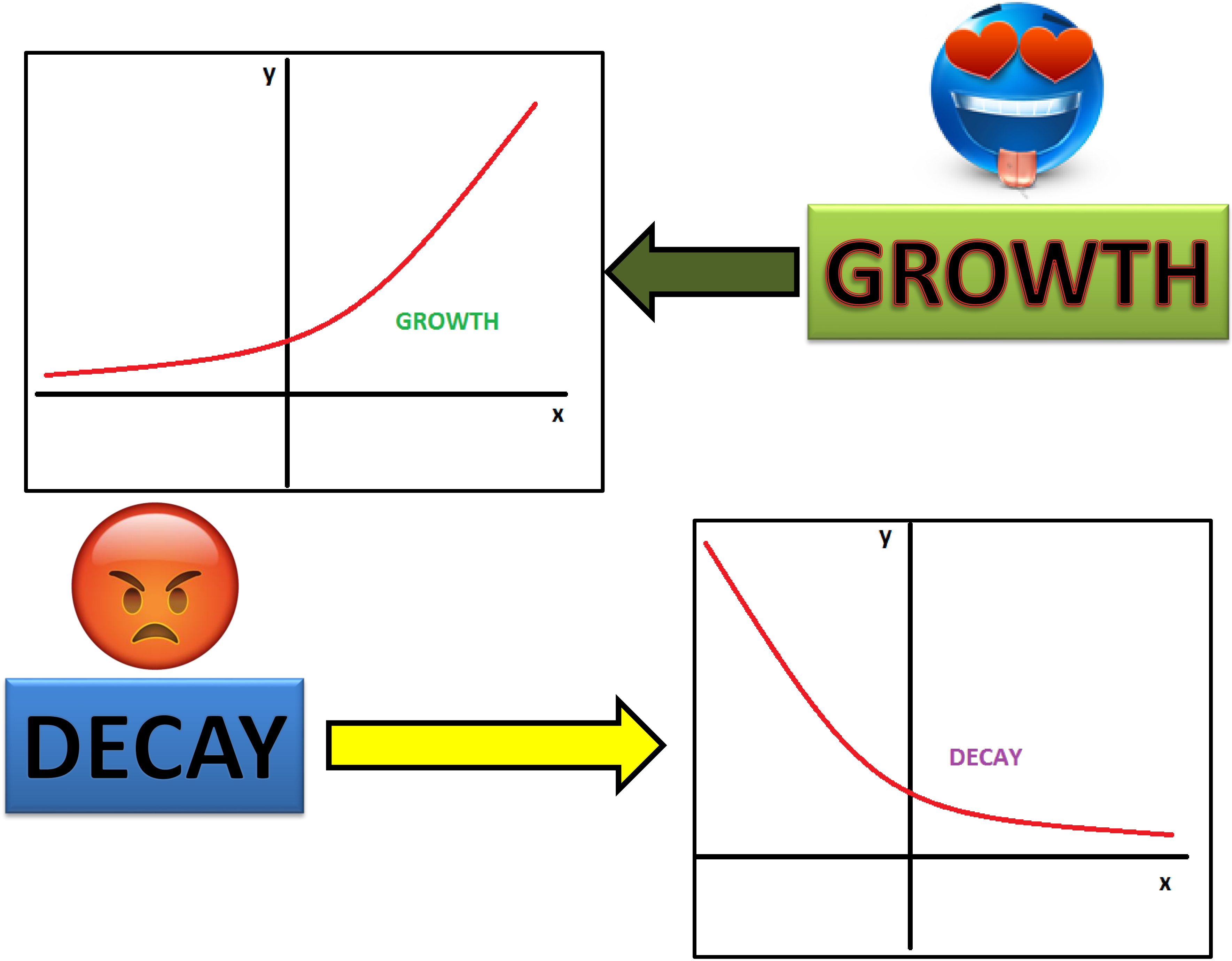
$$y(t) = ke^{at}$$

$a > 0$...GROWING

$a < 0$...DECAYING






GROWTH
y
x
GROWTH
DECAY
y
x
DECAY

Meaning of k & a

The initial value is the value of y at time zero.

$$y(0)\ when\ t = 0$$

$$y_0 = ka^0 = k$$

$$\therefore\ Constant\ k = initial\ value\ of\ y$$

$$y(t) = ka^t$$

$$y(t+1) = ka^{t+1}$$

$$\frac{y(t+1)}{y(t)} = \frac{ka^{t+1}}{ka^t} = a$$

$$\therefore\ y(t+1) = ay(t)$$

Are you ready for a derivation…?

…YES!!

Are you sure…?

Well…check this out…

generally, suppose we know the value of the function at 2 different times …

$$y_1 = ce^{kt_1} \ldots (1)$$
$$y_2 = ce^{kt_2} \ldots (2)$$

dividing the 1st equation by the second equation …

$$\frac{y_1}{y_2} = e^{k(t_1 - t_2)}$$

taking natural log …

$$lny_1 - lny_2 = k(t_1 - t_2)$$

dividng by $t_1 - t_2$ …

$$k = \frac{lny_1 - lny_2}{(t_1 - t_2)}$$

substituting c back …

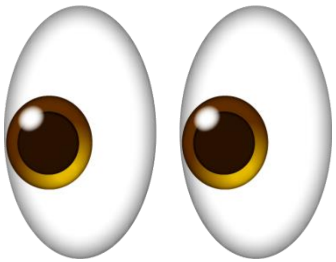
$$y_1 = ce^{\frac{lny_1 - lny_2}{(t_1 - t_2)}.t_1}$$

taking natural log …

$$lny_1 = lnc + \frac{lny_1 - lny_2}{(t_1 - t_2)}.t1$$

in summary we have 2 equations:

$$y_1 = ce^{kt_1} \ldots (1)$$
$$y_2 = ce^{kt_2} \ldots (2)$$



allows us to solve for c …

$$\therefore c = e^{\left(\frac{t_1 . ln y_2 - t_2 ln y_1}{t_1 - t_2}\right)}$$

Shall we solve some examples together…

Example 1:

The mass of a colony of bacteria grows exponentially. Initially the colony has a mass of 2mg.

3 hours later…the mass is 2.1mg.

a) Express the mass of the colony as a function of time.

b) How long does it take for the mass to be 5mg?

$$let\ y(t) be\ the\ mass\ of\ the\ colony\ after\ t\ hours\ ...$$

$$y(0) = 2$$
$$y(3) = 2.1$$

$$y(t) = ka^t$$
$$k = 2$$
$$ka^3 = 2.1$$

$$\frac{2.1}{2} = a^3$$
$$a = \sqrt[3]{1.05}$$
$$a = 1.05^{\frac{1}{3}}$$

$$\boldsymbol{finally}\ ...$$

$$y = ka^t$$
$$= 2\left[(1.05)^{\frac{1}{3}}\right]^t$$
$$= 2(1.05)^{\frac{t}{3}}$$

$$we\ are\ asked\ to\ find\ the\ time\ when\ the\ mass\ is\ 5mg\ ...$$

$$5 = 2(1.05)^{\frac{t}{3}}$$

$$log\frac{5}{2} = log(1.05)^{\frac{t}{3}}$$

$$\frac{t}{3} = \frac{\log 2.5}{log 1.05}$$

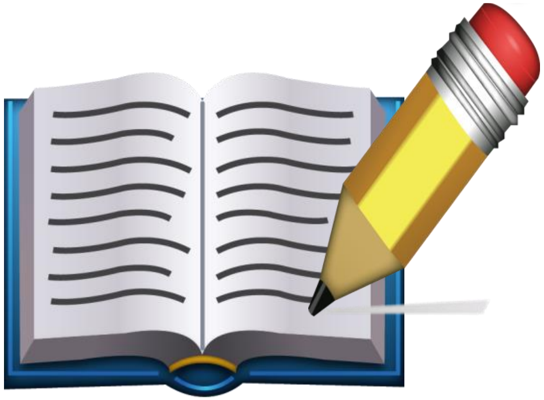
$$t = 3\ \times 18.8$$

$$t = 56.3\ hours$$

Example 2:

The amount of radioactivity given off by a substance decrease by 5% every hour.

After how long will it be 1% of its original value?



$$let\ the\ original\ value\ be\ R$$

$$After\ t\ hours, the\ amount\ will\ be\ R(0.95)^t$$

$$\frac{R}{100} = R(0.95)^t$$

$$log\frac{1}{100} = tlog0.95$$

$$t = -\frac{2}{log0.95}$$

$$t = 89.78\ hours$$

Example 3:

A herd of llamas has 1000 llamas in it.

The population is growing exponentially.

At time t= 4…it has 2000 llamas

Write a formula at any time t.

$$\boldsymbol{f(t) = c.e^{kt}}$$

where $f(t)$ *is the number of llamas at time* t.
c and k are constants to be determined

$at\ t = 0 \ldots there\ are\ 1000\ llamas$
$at\ t = 4 \ldots there\ are\ 2000\ llamas$

$c = value\ of\ f\ at\ t = 0 \ldots which\ is\ 1000$

$$k = \left(\frac{t_1.lny_2 - t_2lny_1}{t_1 - t_2}\right)$$

$$= \frac{ln1000 - ln2000}{0 - 4}$$

$$= ln\frac{1000}{2000} - 4$$

$$= \frac{ln0.5}{-4}$$

$$= \ln\frac{2}{4}$$

$$f(t) = 1000e^{\frac{ln2}{4}}$$

$$f(t) = 1000.2^{\frac{t}{4}}$$

Example 4:

A colony of bacteria is growing exponentially.

At time t = 0…it has 10 bacteria

At time t = 4 it has 2000 bacteria

At what time will it have 100 000 bacteria?

$$k = \left(\frac{t_1 . lny_2 - t_2 lny_1}{t_1 - t_2}\right)$$

$$= \frac{ln10 - ln2000}{0 - 4}$$

$$= \frac{ln\frac{10}{2000}}{-4}$$

$$= \frac{\ln 200}{4}$$

$$f(t) = 10. e^{ln\frac{200}{4}.t}$$

$$= 10. 200^{\frac{t}{4}}$$

$$100\,000 = 10. e^{ln\frac{200}{4}.t} \ldots\ldots. \div 10$$

$$\ln 10\,000 = \ln\frac{200}{4} . t$$

$$t = 4\left(\frac{\ln 10000}{\ln 200}\right) \ldots \ldots. solving\ for\ t$$

$$t = 6.95$$

The half life

Given the process of exponential decay...we can ask how long it would take for the half the original amount to remain...

$$y(t) = \frac{y_0}{2}$$

$$\frac{y_0}{2} = y_0 . e^{-kt}$$

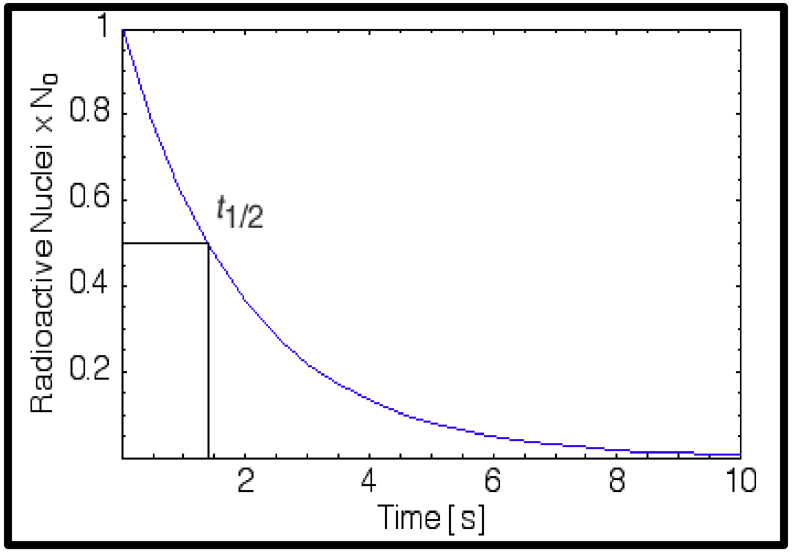
$$\frac{1}{2} = e.^{-kt}$$

taking reciprocals …

$$2 = \frac{1}{e^{-kt}} = e^{kt}$$

$$\ln 2 = \ln e^{kt} = kt$$

$$\boldsymbol{\tau = \frac{ln2}{k}}$$



$\boldsymbol{\tau}$ *this symbol represents half life …!!!*

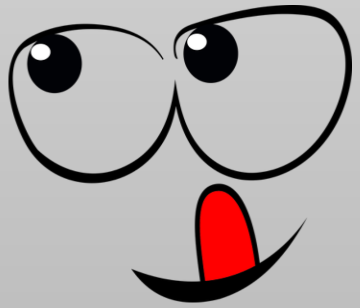

In 1986, the Chernobyl nuclear plant exploded.

Two radioactive elements were released.

These are **Iodine^{131}** whose **half life is 8 days…**

& the other was **Cesium^{137}** whose **half life is 30 years.**

With our model for radioactive decay, we can predict how much of this material would remain over time…



Lets determine the decay constants

$$for\ iodine^{131}$$

$$k = \frac{\ln 2}{\tau}$$

$$= \frac{\ln 2}{8}$$

$$k = 0.0866\ per\ day$$

$$thus\ if\ t\ is\ measured\ in\ days\ \ldots$$
$$the\ amount\ of\ I^{131}\ would\ be\ \ldots$$

$$y(t) = y_0 . e^{-0.0866(t)}$$

$$for\ cesium^{137}$$

$$k = \frac{\ln 2}{\tau}$$

$$= \frac{\ln 2}{30}$$

$$k = 0.023\ per\ year$$

$$y(t) = y_0 e.^{-0.023(t)}$$

how long would I^{131}take to decay to 0.1% of its original value?

$$0.001 y_0 = y_0 . e^{-0.0866(t)}$$

$$0.001 = e^{-0.0866(t)}$$

$$\ln 0.001 = -0.0866\ (t)$$

$$t = 79.7\ days$$