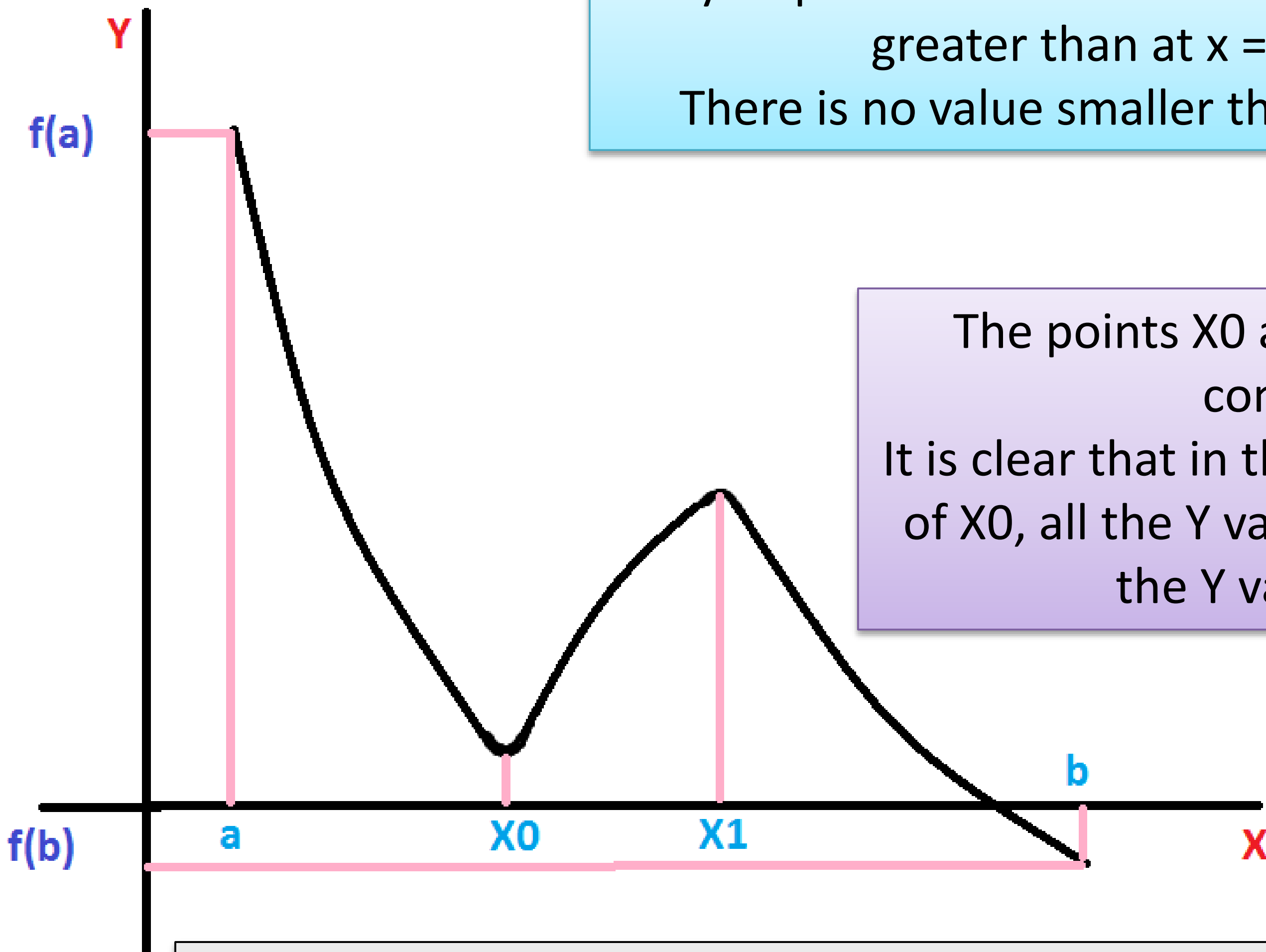
MAXIMA & MINIMA & APPLICATIONS

By inspection we see that there is no Y value greater than at x = a [f(a)]
There is no value smaller than at x = b [f(b)]

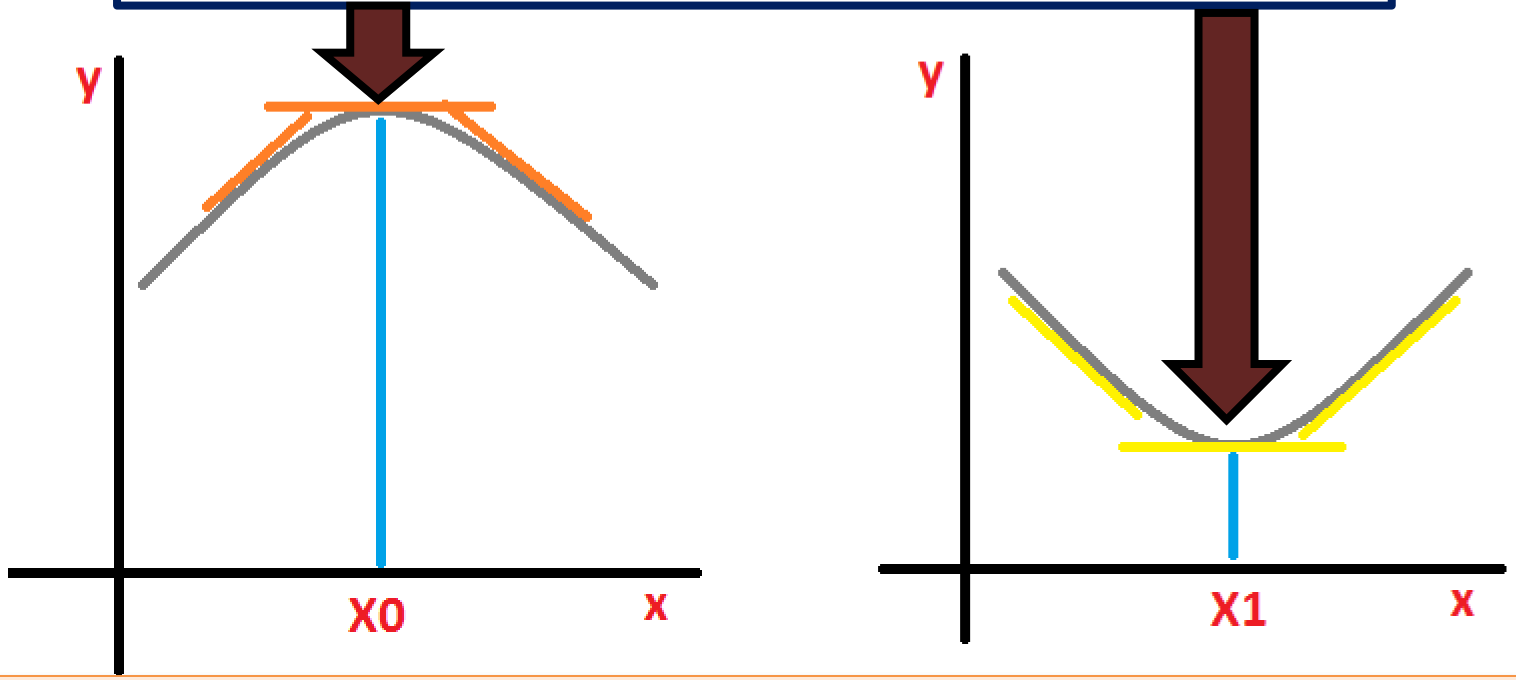


The points X0 and X1 also merit a comment.
It is clear that in the near neighborhood of X0, all the Y values are greater than the Y values at X0.

So f(x) has a **GLOBAL MAXIMUM** at x = a & a **GLOBAL MINIMUM** at x = b
It also has a **LOCAL MAXIMUM** at x = X1 & a **LOCAL MINIMUM** at x = X0

A stationary point on a curve is one at which the derivative has a zero value

STATIONARY POINTS



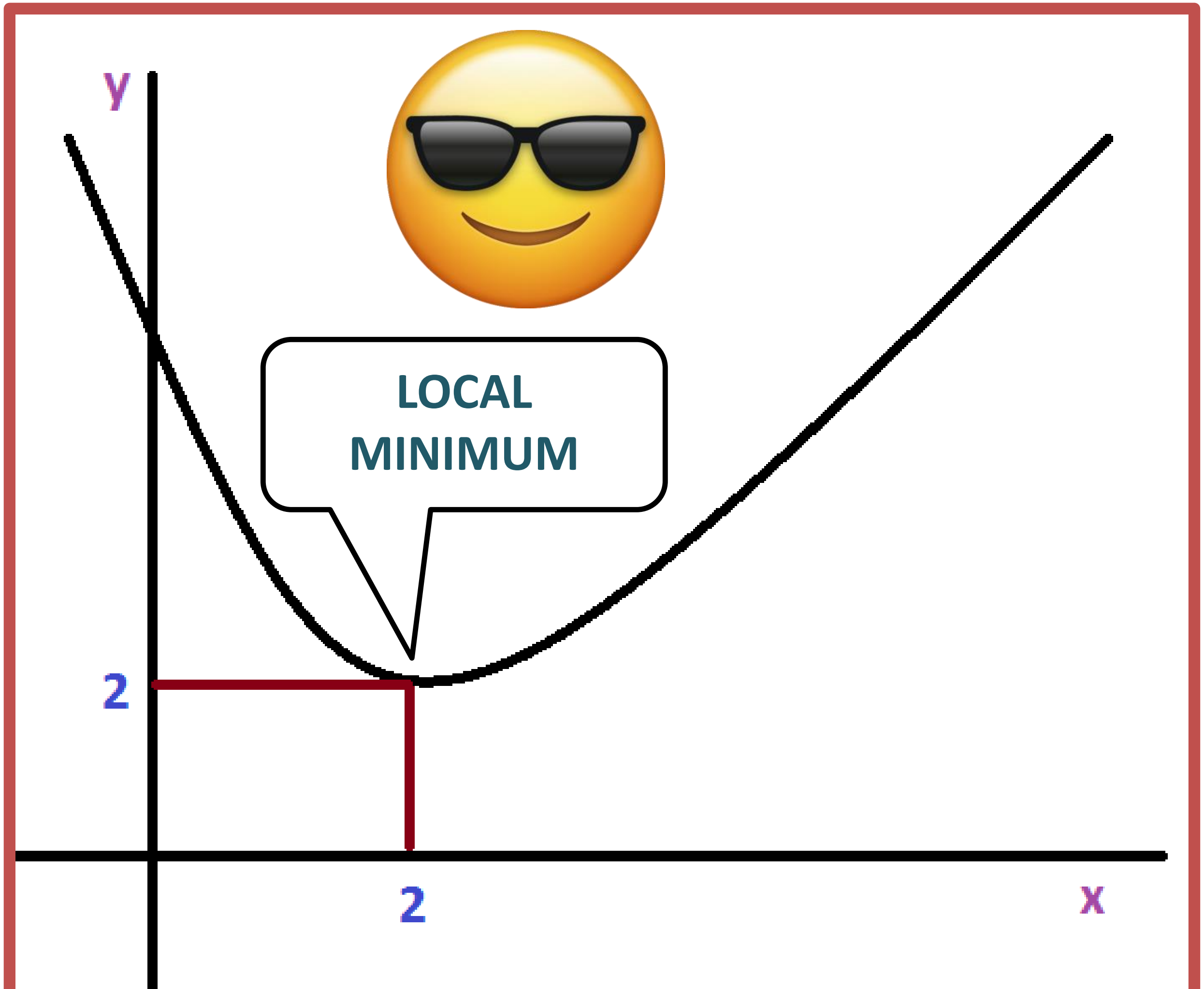
Points on the curve $\boldsymbol{y = f(x)}$ at which $\boldsymbol{\frac{df}{dx} = 0}$ are called **STATIONARY POINTS** of the function

Example:

Sketch the curve

$$y = (x - 2)^2 + 2$$

& locate the stationary points on the curve.



At a stationary point…

$$\frac{df}{dx} = 0$$

We have

$$y = (x-2)^2 + 2$$

So x = 2

The function has 1 stationary point at x = 2 , y = 2.

This point is also called a **LOCAL MINIMUM**

Example:

Find the stationary points:

$$a)\ \ f(x) = 3x^2 + 2x - 9$$

$$b)\ \ f(x) = x^3 - 6x^2 + 9x\ - 2$$

a)

$$f(x) = 3x^2 + 2x - 9$$
$$f'(x) = 6x + 2$$

$$\textit{The stationary points are found by solving the equation} \ldots$$

$$f'(x) = 0$$
$$6x + 2 = 0$$
$$x = -\frac{1}{3}$$

Sub x in $f(x)$

$$f(x) = 3x^2 + 2x - 9$$
$$= 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{3}\right) - 9$$
$$= -\frac{28}{3}$$

b)

$$\boldsymbol{f(x) = x^3 - 6x^2 + 9x - 2}$$
$$f'(x) = 3x^2 - 12x + 9$$

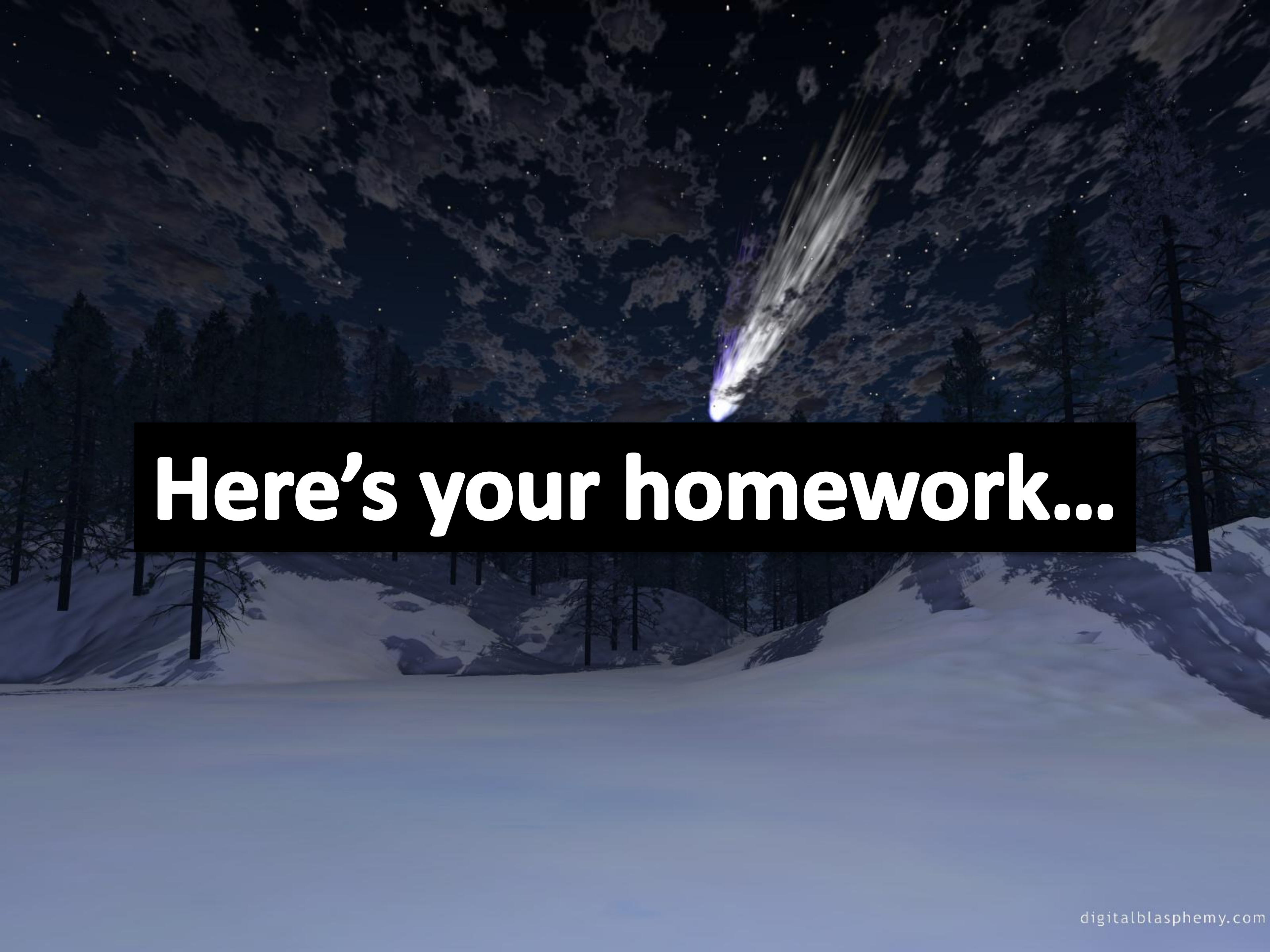
$$3x^2 - 12x + 9 = 0$$
$$= 3(x^2 - 4x + 3)$$
$$= 3(x-3)(x-1)$$
$$x = 3 \ \& \ x = 1$$

Sub x = 3 and x = 1 in $\boldsymbol{f(x)}$

$$3(3)^2 - 12(3) + 9$$
$$\&$$
$$3(1)^2 - 12(1) + 9$$

$$\boldsymbol{stationary\ points\ are\ (3; 2) \& (1; 2)}$$

Here’s your homework…



Exercise 1:

1. $f(x) = -u^2 - 6u + 16$
2. $f(x) = 3x^2 - 4x + 7$
3. $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$
4. $f(x) = x^3 - 6x^2 - 15x + 16$

Find the stationary point of the function

$$y = x^2 - 2x + 3$$

& hence determine the nature of this point.

Determine the nature of the stationary points of the following functions…

$$y = 16 - 6u - u^2$$

$$y = 3x^2 - 4x + 7$$

Find the stationary points of the function

$$y = 2x^3 - 9x^2 + 12x \ - 3$$

& determine their nature

Find the maximum and minimum values of the function

$$y = 4x^3 + 3x^2 - 6x$$

On the interval [-2;1]

APPLICATIONS

Applied Maxima & Minima Problems

The process of finding maximum and minimum values is called **OPTIMIZATION**.

We are trying to do things like maximize the profit in a company, or minimize the costs, or find the least amount of material to make a product.

SHOW ME THE MONEY!!!

Example 1:

The daily profit of an oil refinery is given by:

$$\boldsymbol{P = 8x - 0.02x^2}$$

$$x = number\ of\ barrels\ of\ oil\ refined$$

How many barrels will give the maximum profit?

What is the maximum profit?

$$The\ profit\ is\ a\ maximum\ at\ \frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = -0.04\text{x} + 8$$

$$-0.04x + 8 = 0$$

$$-0.04x = -8$$

$$x = \frac{-8}{-0.04}$$

$$x = 200$$

Is it a maximum?

$$\frac{d^2P}{dx^2} = -0.04 \ldots$$

$$< 0\ thus\ its\ a\ maximum$$

$$When\ x = 200 \ldots$$

$$P = 8(200) - 0.02(200)^2$$

$$P = 800$$

$$If\ the\ company\ refines$$
$$200\ barrels\ a\ day \ldots$$

$$a\ maximum\ profit$$
$$of\ 800\ is\ reached$$

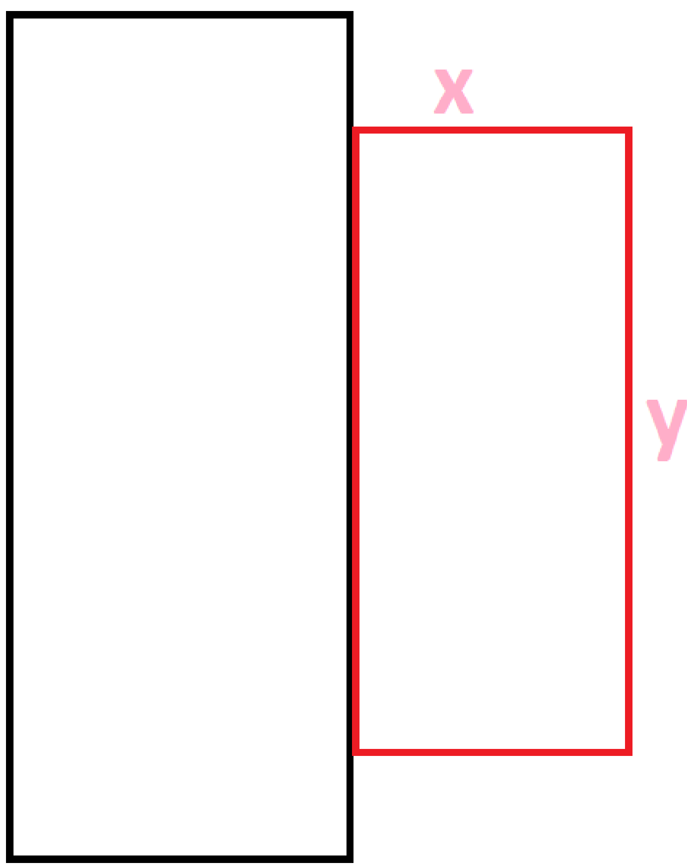
Example 2:

A rectangular storage area is to be constructed along the side of a tall building.

A security fence is required along the remaining 3 sides of the area.

What is the maximum area that can be enclosed with 800m of fencing?

HERE'S WHAT IT LOOKS LIKE…



$$\boldsymbol{Area = xy}$$
$$2x + y = 800$$
$$y = 800 - 2x$$

$$\boldsymbol{Area = xy}$$
$$= x\,(800 - 2x)$$
$$Area = 800x - 2x^2$$

To maximise the area …

$$find\ when\ \frac{dA}{dx} = 0$$

$$\frac{dA}{dx} = 800 - 4x$$
$$800 - 4x = 0$$
$$x = \frac{-800}{-4}$$
$$x = 200$$

$$Substitute\ x = 200\ in\ y$$

$$800 - 2(200)$$
$$= 400$$

Is it a maximum?

$$\frac{d^2A}{dx^2} = -4 \ldots$$
$$< \mathbf{0} \ldots \boldsymbol{Therefore\ maximum}$$

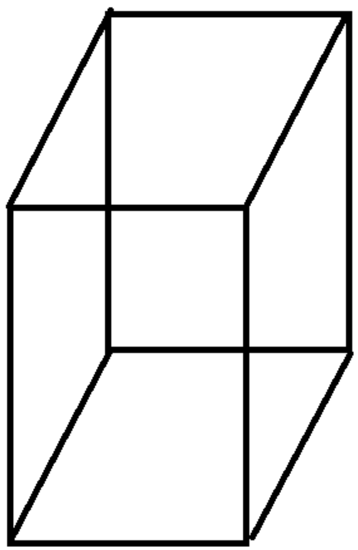
$$Maximum\ area\ occurs\ when$$
$$x = 200\ \&\ y = 400$$

$$Area = 200\ \times 400$$
$$= 80\,000m^2 = 8\ ha$$

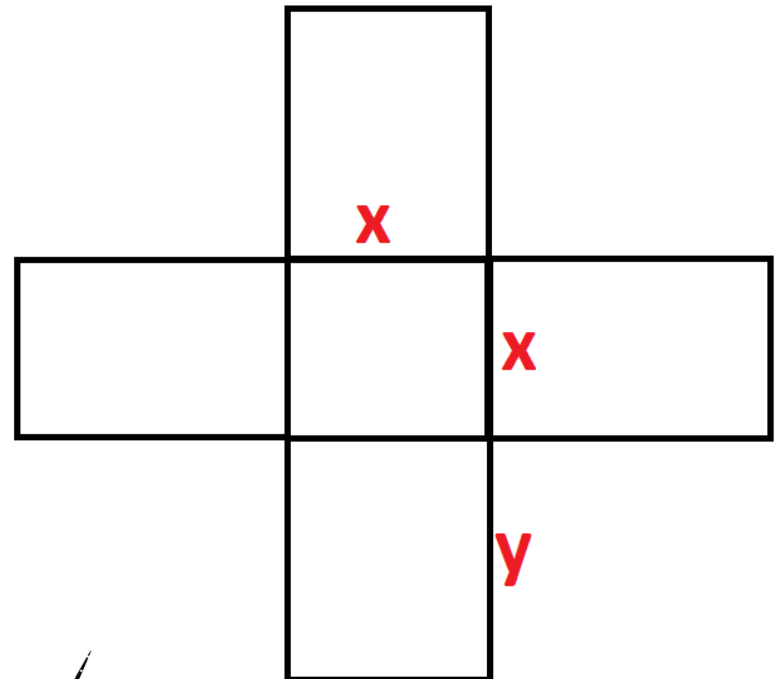
Example 3:

A box with a square base has no top.

If $64cm^2$ of material is used, what is the maximum possible volume for the box?



OPEN IT UP...



$$\boldsymbol{Volume = x^2 y}$$

$$Surface\ area\ of\ the\ box\ is\ 64cm^2$$

$$Area\ of\ the\ base\ of\ the\ box\ is\ x^2$$

$$\&\ area\ of\ each\ side\ is\ xy$$

$$Area\ of\ the\ base + area\ of\ the\ 4\ sides$$

$$is\ given\ by\ \ldots$$

$$\boldsymbol{x^2 + 4xy = 64cm^2}$$

$$4xy = 64 - x^2$$

$$y = \frac{64 - x^2}{4x}$$

$$y = \frac{16}{x} - \frac{x}{4}$$

$$volume\ can\ be$$
$$re-written\ as$$
$$V = x^2y$$
$$= x^2\left(\frac{16}{x} - \frac{x}{4}\right)$$
$$16x - \frac{x^3}{4}$$

$$\frac{dV}{dx} = 16 - \frac{3}{4}x^2$$
$$16 - \frac{3}{4}x^2 = 0$$
$$x = 4.62$$

$$Substitute\ x = 4.62\ in\ y \ldots$$

$$y = \frac{16}{x} - \frac{x}{4}$$
$$= \frac{16}{4.62} - \frac{4.62}{4}$$
$$= 2.31$$

$$is\ it\ a\ maximum?$$
$$\frac{d^2V}{dx^2} = -\frac{3}{2}x \ \ldots . < 0 \ \ldots Maximum$$

$$Maximum\ Volume = x^2y$$
$$= (4.62)^2(2.31)$$
$$= 49.3cm^3$$

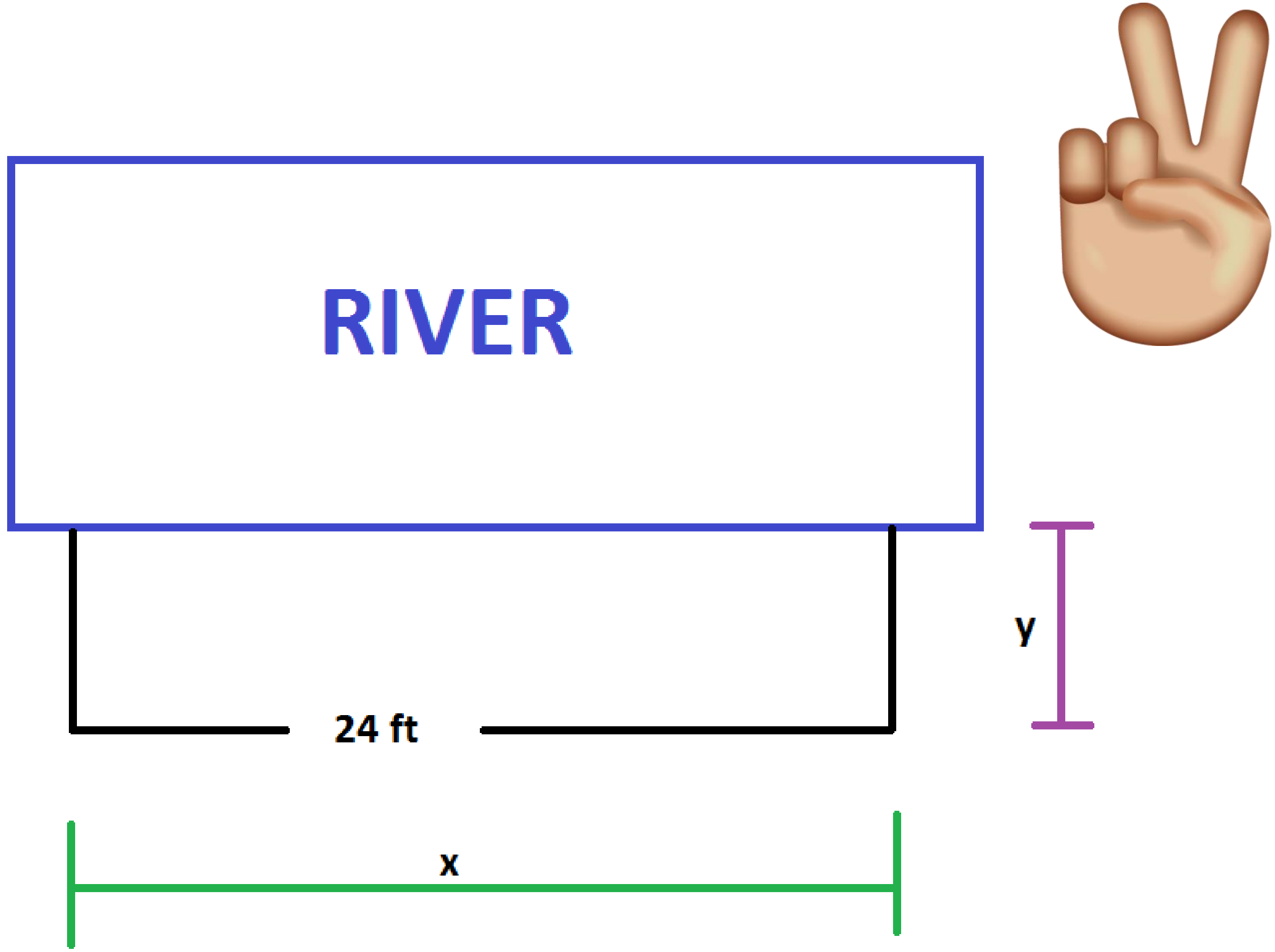
$$Area = x^2 + 4xy$$
$$= (4.62)^2 + 4(4.62)(2.31)$$
$$= 64$$

Example 4:

A rectangular lot is bounded at the back by a river.

No fence is need along the river, and there has to be a 24ft opening at the front.

If the fence along the front costs R1.50 per foot, & along the sides R1.00 per foot, find the dimensions of the largest lot which can be fenced for R300.


RIVER
y
24 ft
x

$$Total\ cost:$$
$$300 = 2y + 1.5(x - 24)$$
$$y = 168 - 0.75x$$

$$\boldsymbol{Area = xy}$$
$$= x(168 - 0.75x)$$
$$= 168x - 0.75x^2$$

$$\frac{dA}{dx} = 168 - 1.5x$$

$$-1.5x + 168 = 0$$

$$-1.5x = -168$$

$$x = \frac{-168}{-1.5}$$

$$x = 112ft$$

$$y = 168 - 0.75(112)$$
$$y = 84ft$$

$$dimensions = 84ft \times 112ft$$